THE

MATHEMATICAL QUESTIONS,

PROPOSED IN THE

LADIES' DIARY,

AND THEIR ORIGINAL ANSWERS,

TOGETHER WITH SOME NEW SOLUTIONS,

FROM ITS COMMENCEMENT IN THE YEAR

1704 TO 1816.

IN FOUR VOLUMES.

BY THOMAS LEYBOURN,

OF THE ROYAL MILITARY COLLEGE.

VOL. I.

London:

PREFACE.

THE Ladies' Diary having been published annually, in the form of an Almanack, for more than a century past; it must, at all times, have been difficult, and, indeed, during the latter part of that period, next to impossible, to obtain a complete copy of the work. This circumstance induced the ingenious and learned gentleman that has superintended it for the last forty-four years (Dr. Charles Hutton, Emeritus Professor of the Mathematics in the Royal Military Academy at Woolwich) to re-publish every thing either curious or valuable, that had appeared in it from its commencement to the year 1773, at which period he became the Editor. Dr. Hutton's edition of the Diaries has, however, been out of print for many years, and is now seldom to be procured; and even if it could be had, forty-four more *Diaries* are now required to complete the work, so that the difficulty of obtaining a perfect set is as great as ever.

On this account, and with a view to gratify such as are anxious to possess what may be considered as a curious, and valuable monument, of the Mathematical genius of the English nation, the Mathematical questions and their answers are here republished, but the

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Enigmas, Rebuses, and such other parts of the work, as have no reference to Mathematics are omitted.

In accomplishing this object, the Editor has availed himself of the labours of Dr. Hutton, by adopting, with his permission, the numerous valuable additions and improvements with which that gentleman had enriched his edition ; and as to the sequel of the work, comprehending the questions and answers since the year 1773, he hopes that, with the aid of some ingenious friends, who have, for a number of years past, contributed to the *Mathematical Repository*, he has been able to render it, in some measure, not altogether unworthy of being incorporated with the labours of his learned predecessor.

Two changes have been made in the disposition of the work, which it is hoped will render it more commodious to the reader. In the first place, each question is now immediately followed by its answer, instead of the questions and answers being separated from one another, as is the case in the original Work, as well as in Dr. Hutton's re-publication ; and in the next place, such questions as had been proposed in the original Diaries in verse, which in almost every case was bad, and often hardly intelligible, are, generally speaking, changed into plain but perspicuous prose.

The importance of the work, as a collection of Practical Exercises in almost every branch of Mathematical learning, must be obvious, if we consider that it is the result of the joint labour of almost all the Mathematicians of eminence, that have appeared in England in the course of the last century. Of these,

passing over such as are yet living, we shall only men-
tion the names of Emerson, Landen, and Simpson,
whose writings, and more especially those of the two
last, have considerably extended the science, both of
pure and mixt Mathematics, and are justly regarded as
among the most elegant examples of analytical investi-
gation *. The work may also be considered as exhibiting
a view of the state of Mathematics in this country, and,
on this account, must be valuable, as illustrative of the
progress of the science in England, during the last
century.

The beneficial influence which the Ladies' Diary, has
exerted upon the state of Mathematical science in this
country, has been long felt, and acknowledged, and has
been particularly noticed by the writer of the very
valuable analysis of the *Mecanique Celeste*, given in the
Edinburgh Review. Speaking of the comparative state
of Mathematical knowledge in England and on the
Continent, he says : — " A certain degree of Mathe-
" matical science, and indeed no inconsiderable degree,
" is perhaps more widely diffused in England than in
" any other country in the world. The Ladies' Diary,
" with several other periodical and popular publica-
" tions of the same kind, are the best proofs of this
" assertion. In these, many curious problems, not of
" the highest order indeed, but still having a con-
" siderable degree of difficulty, and far beyond the
" mere elements of science, are often to be met with :

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* Dr. Maskelyne, the late Astronomer Royal, is known to have
contributed occasionally, under feigned names, to the Diary.

" And the great number of ingenious men who take a-
" share in proposing, and answering these questions,
" whom one has never heard of any where else, is not
" a little surprising. Nothing of the same kind, we
" believe, is to be found in any other country." Again—
" The geometrical part has always been conducted in a
" superior style; the Problems proposed have tended
" to awaken curiosity, and the Solutions to convey
" instruction, in a much better manner than is always
" to be found in more splendid publications."—(See
Edinburgh Rev. vol. xi. p. 282.)

. The Diary was projected and begun in the year
1704 by Mr. John Tipper, who conducted it until
1713. It does not appear that the improvement of
Mathematical Science, was a particular object with the
ingenious projector: indeed the law, which the first
contributors imposed on themselves, of not only pro-
posing, but also of answering all questions in rhyme,
was not favourable to the developement of Mathe-
matical genius.

The superintendence of the work devolved in 1714,
on Mr. Henry Beighton, F. R. S. who, as we learn
from Dr. Hutton's valuable Mathematical Dictionary,
was a respectable mathematician, and mechanist, and
the most eminent civil engineer of his time. He con-
ducted the work until his death, which happened in
1743 or 1744, and gave such satisfaction to the com-
pany of stationers, the proprietors of the work, that
they permitted his widow to continue it, employing a
deputy in the compilation.

The next Editor was Robert Heath, who managed

the Diary from 1745 to 1753: He was a half-pay
Captain of Invalids, and the editor of a similar pe-
riodical work, called the Palladium, as well as some
other works now entirely forgot. He published several
schemes for finding the longitude, and abused the
Astronomer Royal, and Board of Longitude, because
they did not think his productions worthy of notice.
He unfortunately made the Diary a vehicle for low
scurrility; and abused, under various signatures, and in
different publications, all the Mathematical writers of
that day, except Emerson, who is supposed to have
assisted him in his speculations. In particular, he was
virulent beyond all bounds against Mr. Thomas
Simpson, towards whom, he appears to have entertained
great rancour. He, however, in the end, had the worst
of the controversy, for he was expelled from the editor-
ship with disgrace, and succeeded by his rival.

Simpson conducted the work from 1754 to 1760,
and gave it that degree of respectability which it has
ever since maintained. He wrote in it himself, under
various signatures, such as *Hurlothrumbo, Kubernetes,
Patrick O'Cavanah, Anthony Shallow,* Esq. *Timothy
Doodle, Marmaduke Hodgson,* and probably several
others. During this period, his friend, Mr. Landen,
occasionally contributed to the work. Simpson died in
1761, and in the same year, Mr. Edward Rollinson be-
came the Editor, and continued until his death, in
1773. It has ever since been entirely under the able
direction of Dr. Charles Hutton.

Besides the nominal editors, others have, at different
times, superintended this useful publication. In par-

ticular, it has been said, that in the latter years of Mr. Beighton's life, his friend, Mr. Ant. Thacker, who was more skilled in Mathematics, relieved him from the labour of that department.

The Editor would gladly have given some account, of the more distinguished contributors to the Diaries, and have ascribed to the real authors, the solutions they have contributed under feigned names; the information he has gleaned, however, has been so scanty, as to afford but little satisfaction on either of these points. As to biographical notices of English Mathematicians, he believes that all that can possibly be collected of any interest, has found a place in Dr. Hutton's Mathematical Dictionary. There are some writers alive, whose real names would have added dignity to the work, but these, without their consent, he could not take the liberty to insert. The names of such Authors as are dead, and have come to his knowledge, he has inserted in the general Index.

Of the Additional Solutions, which the Editor owes to the kindness of friends, the names of the contributors are put to a part, but not to the whole. He owes, in particular, to his colleagues, the following:

To Dr. Henry Clarke, the additional solution to quest. 206 (vol. i.), marked M.

To Mr. John Lowry, the solutions to which the letters J. L. are annexed.

To Mr. James Cunliffe, those having the letter c.

To Mr. William Wallace, those having the letter w.

To Mr. Mark Noble, those having the letter N, also

the first additional solution to question 51 (vol. i, page 52).

To questions which appeared to be interesting, he has given solutions at considerable length, from academical memoirs, also from the writings of foreign Mathematicians, with a view to the accommodation of mere English readers. Upon the whole, it has been his wish, to provide the Mathematical student with a valuable collection of examples, in every department of the science, to bring together and preserve the labours of many ingenious men, who, during the period of a century, have rendered much service to society, by the practical application of the science in the circumscribed spheres, to which they have been confined, and lastly, to exhibit to the public a picture of the taste of the British nation, for the study of Mathematics, which has certainly more cultivators in this country, than is commonly supposed.

Royal Military College,
November 1817.

MATHEMATICAL QUESTIONS

PROPOSED IN THE

LADIES' DIARY,

AND THEIR ANSWERS.

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Questions proposed in the Diaries for 1707-8, *and answered in those of* 1708-9.

I. QUESTION 1.

In how long time would a million of millions of money be in counting, supposing one hundred pounds to be counted every minute without intermission, and the year to consist of 365 days, 5 hours, 45 minutes?

Answer. 19013 years, 144 days, 5 hours, 55 minutes.

Solution.

The solution of this question is evidently thus: As 100*l*. : 1 minute :: 1000000000000*l*. : 10000000000 minutes = 19013 years 144 days 5 hours 55 minutes, the true time required. H*.

II. QUESTION 2*.

> If to my age there added be
> One half, one third, and three times three;
> Six score and ten the sum you'ld see,
> Pray find out what my age may be.

Answer. 66 years.

Solution.

The meaning of the problem is, that the number 9 added to once his age, together with one half and one third of his age, the sum shall be 130; or since the sum of the parts 1, and $\frac{1}{2}$, and $\frac{1}{3}$ is $\frac{11}{6}$, that $\frac{11}{6}$ of his age is $(130 - 9 =)$ 121; consequently $11 : 6 :: 121 : 66 =$ his age. H.

Algebraic Solution.

Let $6x$ represent the required age; then, by the question, $6x + 3x + 2x + 9$, that is $11x + 9 = 130$; therefore $x = 11$; consequently $6x = 66$, as before. L.

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* All the solutions marked with the signature H, are by Dr. Hutton, and taken, with permission, from his edition of the Ladies' Diaries.

II. QUESTION 2.

A gentleman has a garden of a rectangular form, the length 100 feet and breadth 80, and he wants to make a walk of equal width half way round to take up half the garden: What must be the width of the walk?

Answer. 25·9687576257 feet.

Geometrical Solution.

This problem, I think, may be most elegantly constructed thus: ABCD being the given rectangle, make BE $=$ BA, and CF and CG (on the diagonal) each $=$ CE; then making AH $= \frac{1}{2}$ AF, and AI $= \frac{1}{2}$ AG; and compleating the rectangle AHKI, the thing is done.
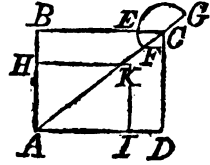
For the area $=$ HA $\times$ AI $= \frac{1}{2}$ FA $\times \frac{1}{2}$ AG $= \dfrac{\text{AC} - (\text{CB} - \text{BA})}{2} \times$

$\dfrac{\text{AC} + (\text{CB} - \text{BA})}{2} = \dfrac{\text{AC}^2 - (\text{CB} - \text{BA})^2}{4} = \dfrac{\text{CB} \times \text{BA}}{2} =$ half the whole area.

And IA $-$ AH $= \dfrac{\text{GA} - \text{AF}}{2} =$ CF $=$ CB $-$ BA, as it ought. H.

The *numerical calculation* will come directly from the construction thus: The required breadth HB is $=$ BA $-$ AH $=$ BA $- \frac{1}{2}$ AF $=$ BA $-$

$\dfrac{\text{AC} - (\text{CB} - \text{BA})}{2} = \dfrac{\text{CB} + \text{BA} - \text{AC}}{2} = \dfrac{100 + 80 - \sqrt{16400}}{2} = 90 -$

$\sqrt{4100} = 25 \cdot 9687576256715.$ H.

The Algebraic Solution.

Put $a =$ CB, $b =$ BA, $d = a - b$, and $z =$ AH. Then HA $\times$ AI

$= z \times (z + d) = \frac{1}{2}ab$; hence $z = \dfrac{\sqrt{(2ab + dd)} - d}{2} =$

$\dfrac{\sqrt{(aa + bb)} - a + b}{2}$, and BH $= b - z = \dfrac{a + b - \sqrt{(aa + bb)}}{2}$

$= \dfrac{\text{CB} + \text{BA} - \text{AC}}{2} =$ half the semi-perimeter minus half the diagonal, the same as above. H.

Another Geometrical Solution.

Let ABCD be the given rectangle. Draw the diagonal AC, and upon DA, BA take DI, BH each equal to $\frac{1}{2}$ (BC + BA $-$ AC); through I and H draw IK, HK respectively parallel to DC, BC, to meet in K and the thing is done.

By construction $\frac{1}{2}$(BC + BA $-$ AC) $=$ BH, that is BC + BA $=$ AC + 2BH, and BC^2 + BA^2 + 2BC $\times$ BA $=$ AC^2 + 2BC $\times$ BA $=$ AC^2 + 4AC $\times$ BH + 4BH^2, or 2BC $\times$ BA $=$ 4AC $\times$ BH + 4BH$^2 =$ 4BH (AC + BH); whence $\frac{1}{2}$BC $\times$ BA $=$ BH $\times$ (AC + BH). But AC + 2BH $=$ BC + BA, that is AC + BH $=$ BC + BA $-$ BH $=$ BC + HA; therefore $\frac{1}{2}$BC $\times$ BA $=$ BH (AC + BH) $=$ BH (BC + HA). Now

$\frac{1}{2}$BC × BA expresses half the area of the rectangle ABCD, and BH (BC + HA) expresses the area of the space occupied by the walk; therefore the walk is half the garden. L.

III. QUESTION 3.

If thirteen tuns of claret cost nineteen pounds, how many pints can be had for a thousand crowns?

Answer. 344842 $\frac{2}{19}$ pints.

Solution.

As 19l. : 26208 pints ∷ 13 tuns :: 250l. ∷ 1000 crowns : 344842$\frac{2}{19}$ pints. H.

IV. QUESTION 4.

If thirteen marks with fourteen groats will buy fifteen loads of hay, how many pounds together with sixteen crowns will pay for ninety loads?

Answer. 49l. 8s.

Solution.

As 15 loads : 90 loads :: 1 : 6 :: 178s. ∷ 13 marks + 14 groats : 1068s. ∷ the value of 90 loads. From which deducting 80s. ∷ 16 crowns, leaves 988s. ∷ 49l. 8s. or 49$\frac{2}{5}$l. for the number required. H.

V. QUESTION 5.

A person remarked that upon his wedding day the proportion of his own age to that of his wife was as three to 1; but 15 years afterwards the proportion of their ages was as 2 to 1. What were their ages upon the day of marriage?

Answer. The husband's age was 45, and the wife's 15 years.

Solution.

Put z and $3z$ for the two ages, and $a = 15$; then $2 : 1 :: 3z + a : z + a$, or $3z + a = 2z + 2a$; hence $z = a = 15$, and $3z = 45$.

But perhaps the more masterly way of solving this problem is thus: since $2 : 1 :: 3z + a : z + a$, subtract each consequent from its antecedent, and we have $1 : 1 :: 2z : z + a$;

but $1 : 1 :: z : z$,   hence,

by subtracting, $1 : 1 :: z : a$; or $z = a =$ the number added universally. H.

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Questions proposed in 1709, and answered in 1710.

I. QUESTION 6.

A rich Indian merchant brought sixty-four diamonds of an extraordinary value and great beauty to sell to a King of Persia, and

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upon being asked the price of some said, that he would sell them all together upon the following conditions, viz. for the first stone he would have a grain of wheat, for the second stone two grains, for the third stone four grains, &c. doubling the number of grains for each succeeding stone. The King thinking it an advantageous bargain, readily agreed to the proposal, imagining that a few sacks of wheat, at the most, would pay for the whole. But how great was his astonishment, when informed that all the corn in his kingdom would be far short of satisfying the merchant's demand. Now suppose a pint of corn to contain 10000 grains, and a bushel to weigh half a *cwt.*; what would the whole quantity of wheat be worth at 5s. per bushel, and how many horses carrying 1000l. each would it take to remove the same: also how many ships of 100 tons burden, would it load?

Solution by a Lady.

Suppose (saith she) navigation to have begun with Adam, and 67279 ships of 100 tons each to have been built every year since the creation of the world (which she supposes to be about 5655 years); and so to build the same number continually to the end of the world, and the world to last 5000 years longer, and all those ships to be in being at once; all of them together, according to the tenor of the question, would not be sufficient to transport the wheat, so prodigious is the quantity. She adds,

For there would be 7205759403 horse loads, (besides 7921. 15s. 10d. odd money) and the same number of ship loads.

Additional Solution.

The number of terms being 64, the ratio 2, and the first term 1; the sum of all the terms of the series will be $\dfrac{2^{64}-1}{2-1} \times 1 = 2^{64} - 1$

$= 18446744073709551615$ for the number of grains of wheat; which being divided by $10000 \times 64 = 640000$ the number of grains in a bushel, we have $28823037615171 \cdot 1743984375$ for the number of bushels; the value of which at 5s. each is $7205759403792 \cdot 79$, &c. l. and the double of this number will be the weight in *cwts.* then the value divided by 1000, and the weight by 2000, give each $7205759403 \cdot 79$ &c. for the number of horse or ship loads required. **H.**

II.   QUESTION 7.

Two persons, of the same age, discoursing together, says one to the other, my eldest son is just half as old as I am, and if $340\frac{1}{2}$ is added to the continual product of our three ages, the sum will amount to 30000; required the ages of all three.

Answer. The age of the son is 19 years and a half, and that of each of the two persons 39 years.

Solution.

This question is to find three numbers, of which two shall be equal to each other, and each double of the third, and whose continual product shall be (29659·5) 340·5 less than 30000 : or, to find two numbers, the one double the other, and the less multiplied by the square of the greater shall produce 29659·5 : or, to find a number, the half of whose cube shall be 29659·5, : or its whole cube 59319 ; and consequently that number must be $\sqrt[3]{59319} = 39 =$ the age of each of the elder ; and consequently $19\frac{1}{2}$ is that of the youngest.   H.

Algebraical Solution.

Let x represent the age of the son, then $2x$ will represent the age of each of the elder.   By the question $2x \times 2x \times x + 340\cdot5 = 30000$; therefore $4x^3 = 29659\cdot5$; $x^3 = 7414\cdot875$, and $x = 19\cdot5$ the age of the son ; and $2x = 39$ years the age of each of the elder persons.   L.

III.   QUESTION 8.

A Vintner has wines of 8, 5 and 4 groats per quart, and wants to make a mixture of 56 quarts, worth 22 pence a quart; how many different ways can this be done in whole numbers?

Answer.

Best   10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 quarts, at 32d. each.
Mean 44, 40, 36, 32, 28, 24, 20, 16, 12, 8, 4 quarts, at 20d. each.
Worst 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32 quarts, at 16d. each.
Being eleven answers in whole numbers, which are all that can be found to sell 55 quarts, at 22d. a quart.

Solution.

Putting x, y, and z for the number of quarts of the best, middle, and worst sort, respectively ; then $x + y + z = 56$, and $32 x + 20y + 16 z = 22 \times 56 = 1232$; hence $z = 3x - 28$, and $y = 84 - 4x$. Now x must be such a whole number as will make both $3x - 28$ and $(84 - 4x$ or$)$ $21 - x$ whole numbers ; from the latter of these two equations, it appears that the greatest value of x is 20 ; and from the former, that its least value is 10 ; so that x may be any number from 10 to 20 inclusive ; and by substituting every number from 10 to 20 for x, in the two equations $z = 3x - 28$, $y = 84 - 4x$, we obtain these eleven answers in whole numbers, viz.
$x = $ 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 quarts, at 32d. each.
$y = $ 44, 40, 36, 32, 28, 24, 20, 16, 12, 8, 4 quarts, at 20d. each.
$z = $ 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32 quarts, at 16d. each.
                                                                                M.

IV.   QUESTION 9.

Seven men bought a grinding stone, the diameter of which was 5 feet, and they agreed that each should use it till he had ground away his share.   What part of the diameter must each grind away?

Answer. The first must grind 4·4508 inches.

           Second .... 4·8400
           Third .... 5·3535
           Fourth .... 6·0765
           Fifth .... 7·2079
           Sixth .... 9·3935
           Seventh .... 22·6778

                      60 inches.

Solution.

This question is to divide a circle, of 60 inches diameter, into 7 equal parts, or rings, bounded by concentric circles; of which the solution will be thus :—The whole circle, and each inner circle, after the several preceding rings are ground off, must be to each other, by the question, as the numbers 7, 6, 5, 4, 3, 2, 1; but circles are as the squares of their diameters; therefore the diameters of those circles will be to one another as $\sqrt{7}, \sqrt{6}, \sqrt{5}, \sqrt{4}, \sqrt{3}, \sqrt{2}, \sqrt{1}$; but the greatest diameter is 60 or $60\sqrt{\frac{7}{7}}$; therefore, by proportioning, all the other diameters will come out thus, $60\sqrt{\frac{6}{7}}, 60\sqrt{\frac{5}{7}}, 60\sqrt{\frac{4}{7}}, 60\sqrt{\frac{3}{7}}, 60\sqrt{\frac{2}{7}}, 60\sqrt{\frac{1}{7}}$. Now the last of these is the diameter of the last person's share, and the difference between every two adjacent terms being taken, will give the double breadth of the rings, or the parts of the whole diameter to be ground off by the other persons, viz.

$$60\sqrt{\tfrac{7}{7}} - 60\sqrt{\tfrac{6}{7}} = \frac{\sqrt{7} - \sqrt{6}}{\sqrt{7}} \times 60 = 4·450794 \text{ the 1st person's share.}$$

$$60\sqrt{\tfrac{6}{7}} - 60\sqrt{\tfrac{5}{7}} = \frac{\sqrt{6} - \sqrt{5}}{\sqrt{7}} \times 60 = 4·839951 \text{ the second}$$

$$60\sqrt{\tfrac{5}{7}} - 60\sqrt{\tfrac{4}{7}} = \frac{\sqrt{5} - \sqrt{4}}{\sqrt{7}} \times 60 = 5·353518 \text{ the third}$$

$$60\sqrt{\tfrac{4}{7}} - 60\sqrt{\tfrac{3}{7}} = \frac{\sqrt{4} - \sqrt{3}}{\sqrt{7}} \times 60 = 6·076516 \text{ the fourth}$$

$$60\sqrt{\tfrac{3}{7}} - 60\sqrt{\tfrac{2}{7}} = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{7}} \times 60 = 7·207871 \text{ the fifth}$$

$$60\sqrt{\tfrac{2}{7}} - 60\sqrt{\tfrac{1}{7}} = \frac{\sqrt{2} - \sqrt{1}}{\sqrt{7}} \times 60 = 9·393480 \text{ the sixth}$$

$$60\sqrt{\tfrac{1}{7}} - 60\sqrt{\tfrac{0}{7}} = \frac{1}{\sqrt{7}} \times 60 = 22·677870 \text{ the seventh}$$

           the sum of them all is 60·000000 the whole
                             diameter.

The Construction will be thus : Divide the radius AB of the given circle into 7 equal parts; and at the points of division erect perpendiculars meeting the circle, described on the diameter, AB, in D, F, H, K, M, O; then with the centre A, and radiuses AO, AM,
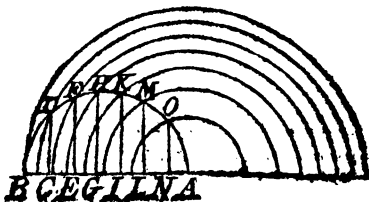
AK, &c. circles being described, the thing is done. For, by the nature of the circle, the squares of the chords or radiuses AO, AM, AK, &c. are as the versed sines AN, AL, AI, &c.

SCHOLIUM. It is evident that the above method of calculation and construction will both hold true also when the shares are unequal in any proportion; by using the respective proportional numbers in the former, and dividing the radius AB in the same proportion in the latter. H.

The above solution has been inserted by Dr. Hutton, at the end of a valuable work which he has just published, entitled "Tracts on Mathematical and Philosophical Subjects; comprising, among other important articles, the Theory of Bridges, with several plans of recent improvement. Also the results of numerous experiments on the force of Gunpowder, with applications to the modern practice of Artillery;" in 3 vols. 8vo. There is a short history of the Problem and some anecdotes connected with it, prefixed to the solution, which we shall here insert, with the author's permission, believing that they will not fail to prove interesting to the reader.

"A particular case of the present problem was first of all, " as far as I know, proposed in that useful and valuable little " annual work, the Ladies' Diary, for the year 1709. In the " year following an answer was so far given only, as merely to " specify the numbers denoting the parts of the diameter to be " ground down, or cut off, without any mode of solution whatever, " either arithmetical or geometrical. Many years after, a geome- " trical construction was given by a Mr. Hawney, in his little book " on Mensuration, but so clumsy in its manner, as to require the " description of a separate circle to ascertain the point through " which each of the dividing concentric circles was to pass. And in " this state it remained till about the year 1770, when Mr. James " Ferguson, the ingenious lecturer on astronomy and mechanics, in " his peregrinations came to Newcastle, where I then resided, to " give the usual course of his public lectures; on which occasion, " with the assistance of my friends, I not only procured him a nu- " merous and respectable audience, but also accommodated him " with the free use of the new school-rooms, which I had lately " built, to deliver his lectures in. As Mr. F. commonly amused my " family and friends at evenings, with showing his ingenious me- " chanical contrivances and drawings, on one of these occasions he " produced a very neat and correct drawing, on a large scale, being " a construction of this problem, in the very prolix way as before " given by Hawney; but which he exhibited as a great curiosity. " I ventured to remark to him that I thought a much simpler con- " struction might be found out, for this problem, which was then " new to me. As Mr. F. expressed a wish to see such a thing as " a simpler construction, which however he seemed to have his " doubts of procuring, I was induced to consider it that evening, before " going to rest, and discovered the construction as given above.

" The next morning I showed him the new and very simple con- " struction, with its demonstration, which he seemed much pleased

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" with, on account of the apparent simplicity, but doubted very
" much that it might not be correctly true. On referring him to the
" accompanying demonstration, to satisfy himself of its geometrical
" truth, I was much surprised by his reply, that he could not un-
" derstand that, but he would make the drawing correctly on a large
" scale which was always his way to try if such things were true.
" In my surprise I asked where he had learned geometry, and
" by what Euclid or other book; to which he frankly replied he
" had never learned any geometry, nor could ever understand the
" demonstration of any one of Euclid's propositions. Accordingly
" the next morning, with a joyful countenance, he brought me the
" construction, neatly drawn out on a large sheet of paste-board,
" saying he esteemed it a treasure, having found it quite right, as
" every point and line agreed to a hair's breadth, by measurement
" on the scale. This problem and the construction he afterwards
" inserted, with the proper acknowledgment, as a curiosity, in his
" Select Mechanical Exercises, page 123. Printed in 1773.

" Very soon after the first publication of the solution of the Diary
" question as above, it was seized by a shameful plagiary, in the
" person of a Mr. Samuel Clark, who had commenced, after mine, a
" re-publication of the Diary questions, under the title of ' The
" Diarian Repository.' When this editor arrived at the question
" above referred to, he copied my solution, and very inconsistently
" ascribed it to Mr. Hawney before mentioned, though it was mani-
" fest to every geometrician that no two constructions could be more
" unlike one another, as the one employed seven different circles, and
" as many separate constructions, in effecting what the other ac-
" complished by only one single circle and construction. From the
" many gross errors, and the numerous omissions and absurdities in
" that ' Repository of Errors,' as it was commonly called, it soon
" fell under the necessity of coming to a sudden and premature end."

V. QUESTION 10.

What is the length of the pendulum which will make as many
vibrations in a minute as it is inches in length?

Answer. 52·063039916251851082, being the cube root to 20
decimals.

Solution.

Since the lengths of different pendulums are to each other reci-
procally as the squares of their vibrations made in the same time;
we shall have, putting x for the length required, $x : 39·2 :: 60^2 = 3600 : xx$; hence $x^3 = 3600 \times 39·2$, and $x = \sqrt[3]{3600 \times 39·2} = \sqrt[3]{141120} = 52·06304$ inches, nearly. H.

Questions proposed in 1710, *and answered in* 1711.

I.   QUESTION 11.

A man being asked what money he possessed, replied that he had three sorts of coin, namely, half crowns, shillings, and sixpences; the shillings and sixpences together, amounted to 409 pieces; the shillings and half-crowns to 1254 pieces; but if 42 was subtracted from the sum of the half-crowns and sixpences, there would remain 1103 pieces.   What sum did the man possess in all?

Answered by Mr. Thomas Gosling.

995 half-crowns, 259 shillings, 150 sixpences; in all 141*l.* 1*s.* 6*d.*

Solution.

The meaning of this question is, that the sum of the shillings and sixpences is 409, the sum of the shillings and half-crowns is 1254, and the sum of the sixpences and half-crowns 1145; to find the number of each.   That is, having given the sums of every two of three numbers, to find the numbers.

Put $x =$ the number of sixpences, $y =$ the shillings, and $z =$ the half-crowns.   Then, by the question, $x + y = 409$, $y + z, = 1254$, and $x + z = 1145$.   The sum of the first and third, is $2x + y + z = 1554$, from which taking the second, we have $2x = 300$; hence $x = 150$, $y = 409 - x = 259$, and $z = 1145 - x = 995$.   And the sum of these all together amounts to 141*l.* 1*s.* 6*d.*   H.

Otherwise.

Put $a = 409$, $b = 1254$, and $c = 1145$.   Then by the question,

$$x + y = a,$$
$$x + z = b,$$
$$y + z = c.$$

The sum of the three equations is

$$2x + 2y + 2z = a + b + c,$$
$$\text{or}\quad x + y + z = \frac{a + b + c}{2}.$$

From this equation, take each of the given ones; then

$$x = \frac{a + b + c}{2} - c = \frac{a + b - c}{2}$$

$$y = \frac{a + b + c}{2} - b = \frac{a - b + c}{2}$$

$$z = \frac{a + b + c}{2} - a = \frac{b + c - a}{2}.$$

In this manner the problem may be resolved, whatever be the number of equations.   Thus, suppose it were required to find four numbers having given the sum of every three.

Let x, y, z and v represent the four numbers required, and a, b, c, d the four given sums. Then

$$x + y + z = a,$$
$$x + z + v = b,$$
$$x + y + v = c,$$
$$y + z + v = d.$$

Take the sum of all the equations and divide by three, we have

$$x + y + z + v = \frac{a + b + c + d}{3}.$$

From this equation take each of the given ones, and

$$x = \frac{a + b + c + d}{3} - d = \frac{a + b + c - 2d}{3},$$

$$y = \frac{a + b + c + d}{3} - b = \frac{a + c + d - 2b}{3},$$

$$z = \frac{a + b + c + d}{3} - c = \frac{a + b + d - 2c}{3},$$

$$v = \frac{a + b + c + d}{3} - a = \frac{b + c + d - 2a}{3}.$$

And in general if there were n unknown quantities, of which the sum of every $(n - 1)$ was given, then the quantities might be found in a similar manner by subtracting each of the given sums from the aggregate of all the sums, divided by $n - 1$.   E.

II. QUESTION 12.

A gentleman has to pay a thousand pounds, and has only two sorts of coin, guineas valued at 21s. 6d. each, and louis-d'ors at 17s. each; how many different ways may the payment be made with these coins.

Answered by Mr. John Boswel.

Guineas—32, 66, 100, 134, &c. by adding always 34.
Louis-d'ors—1136, 1093, 1050, 1007, &c. by subtracting always 43
   will give 27 answers, each two sums making 1000l.

Solution.

Put x for the number of guineas, and y for that of the louis-d'ors. Then will $43x + 34y$ be $= (1000 \times 40 =) 40000$; hence $y = \dfrac{40000 - 43x}{34} = $ (by dividing) $1176 - x - \dfrac{9x - 16}{34}$; consequently $\dfrac{9x - 16}{34} = $ a whole number, as also 4 times the same, viz. $\dfrac{36x - 64}{34} = \dfrac{18x - 32}{17} = $ (by dividing) $x - 2 + \dfrac{x + 2}{17}$; hence $\dfrac{x + 2}{17} = $ a whole number $= p$; then $x = 17p - 2$. This written for it in the ex-

pression $y = \dfrac{40000 - 43x}{34}$, gives $y = 1179 - 21\frac{1}{2}p$; consequently

$p =$ any even number $=$ (suppose) $2n$; then $y = 1179 - 43n$, and $x = 34n - 2$; where n is any integer number.

Taking successively the numbers 1, 2, 3, 4, &c. for n, we have
$x =$ 32,   66,   100,   134, 168, 202, 236, 270, 304, &c.
$y =$ 1136, 1093, 1050, 1007, 964, 921, 878, 835, 792, &c.   H.

III.   QUESTION 13.

A farmer hired a servant 30 days, upon the condition, that for every day he worked he was to receive sixteenpence, and for every day he was idle or absented himself he should forfeit twenty-pence; now upon coming to settle their account it was discovered that the farmer had nothing to pay, and the servant nothing to receive; how many days did the servant work, and how many was he idle?

Answer.   He worked 16 days 8 hours, and idled 13 days 4 hours.

Solution.

Since the two products of each number of days by their respective prices are equal to each other, it follows that the said number of days will be reciprocally as the prices; but the prices are as 20 to 16, or as 5 to 4, whose sum is 9, and the whole number of days is 30; whence 9 : 30, or as 3 : 10 :: $\begin{cases} 5 : 16\frac{2}{3} \text{ days worked.} \\ 4 : 13\frac{1}{3} \text{ days idled.} \end{cases}$ H.

Algebraic Solution.

Let $x =$ the number of days he worked, and $y =$ those he was idle.
Then by the question $x + y = 30$,
and $16x = 20y$.

The first equation gives $x = 30 - y$, and the second gives $x = \dfrac{20y}{16} = \dfrac{5y}{4}$.

Therefore $30 - y = \dfrac{5y}{4}$, $\therefore$ $120 - 4y = 5y$, $\therefore$ $120 = 9y$, $\therefore y = \dfrac{120}{9} = 13\frac{1}{3}$.

Consequently he worked $16\frac{2}{3}$ days, and idled $13\frac{1}{3}$ days.   L.

IV.   QUESTION 14.

How long must the tether of a horse be to allow him just to graze an acre?

Answered by Mr. Nath. Browne.

The length of the tether is 39·2507375 yards.

Solution.

The number of square yards in an acre being 4840, and a circle being equal to 3·14159 &c. drawn into the square of the radius;

therefore $\dfrac{4840}{3\cdot14159 \ \&c.} =$ the square of the radius, and conse-

quently the radius itself or length of the cord will be $\sqrt{\dfrac{4840}{3\cdot14159 \ \&c.}}$

$= 39\cdot25073$ yards. H.

V. QUESTION 15.

A lady being asked her age by an impertinent spark, made the following reply; if my age be multiplied by three, and two-sevenths of that product trebled, the square root of two-ninths of that will be four. Now tell my age or never see me more?

Answered by Mr. John Ford.

The Lady's age was twenty-eight years.

Solution.

Put x for the lady's age. Then, by the question, $\sqrt{(3x \times \frac{2}{7} \times 3 \times \frac{2}{9})}$

$= \sqrt{\dfrac{36x}{63}} = 4$; hence $x = \dfrac{16 \times 63}{36} = \dfrac{16 \times 7}{4} = 4 \times 7 = 28.$ H.

VI. PRIZE QUESTION 16.

Walking through Cheapside, London, on the first day of May, 1709, the sun shining brightly, I was desirous to know the height of Bow steeple. I accordingly measured its shadow just as the clock was striking twelve, and found its length to be $253\frac{1}{8}$ feet; it is required from thence to find the steeple's height.

Answered by Mrs. Mary Wright.

May 1, 1709.   o   /   //

Sun's longitude, from its ingress into aries..............51 28   0
Oblique angle of the ecliptic and equator,..............23 29   0
Thence the declination that day.........................18   9 45
Consequently its merid. altitude in lat. 51° 32'..........56 37 45
The complement thereof to 90 is........................33 22 15

Then as the sine of the angle 33° 22′ 15″.
To the base 253·125 feet.
So is the sine of the angle 56° 37′ 45″.
To the perpendicular 384·307 feet the height of the steeple.

Note. The true height of Bow steeple is 225 feet, for which at first I had proportioned the length of the shadow, but upon second thoughts I altered it, for fear some, who had read its height in history, should claim the reward, without having art enough to investigate it by trigonometry.

Solution.

This solution, at length will be thus :

The sun's longitude being supposed 51° 28′, and the angle made by the equator and ecliptic 23° 29′ ; by right-angled spherical triangles, as radius : sine of 51° 28′ :: sine 23° 29′ : sine 18°, 9′ 45″ the declination for the time ; to which adding 38° 28′ the complement of the latitude, there results 56° 37′ 45″ for the meridian altitude that day. Then, by right-angled plane triangles, the height of the steeple will be found by Mrs. Wright's proportion above, or rather thus, as radius : 56° 37′ 45″ :: 253½ : 384·31 the steeple's height. н.

——⊸⊕✕⊕⊸——

Questions proposed in 1711, and answered in 1712.

I. QUESTION 17.

I happen'd one ev'ning with a tinker to sit,
Whose tongue ran a great deal too fast for his wit:
He talk'd of his art with abundance of mettle,
So I ask'd him to make me a flat-bottom kettle,
That the top and the bottom diameters be
In just such proportion as five is to three.
Twelve inches the depth I would have and no more,
And to hold in ale gallons seven less than a score.
He promis'd to do't, and to work he strait went;
But when he had done it, he found it too scant.
He alter'd it then, and too big he had made it,
And when it held right, the diameters fail'd it:
So that making't so often, too big, or too little,
The tinker at last had quite spoil'd the kettle:
Yet he vows he will bring his said purpose to pass,
Or he'll utterly spoil ev'ry ounce of his brass.
To prevent him from ruin, I pray help him out,
The diameter's length else he'll never find out.

Answered by Mr. Richard Parker.

Lesser diameter 14·6390238 ; greater diameter 24·398373 inches.

First Solution.

Putting $5z$ and $3z$ for the top and bottom diameters, by page 525 Mensuration, we shall have $(3 \times 64zz + 4zz$ or$)$ $196zz \times$

$12 \times ·00023209 = 13$: hence $z = \sqrt{\dfrac{13}{196 \times 12 \times ·00023209}}$

$= 4.880057$; consequently $3z = 14.640171$, and $5z = 24.400285$, the two diameters required. H.

Second Solution.

Find the content of the frustum of a cone, whose altitude is the same as that of the given one, and the diameters of the top and bottom 5 and 3. This content will (by prob. 6. vol. ii. page 48. of the course) be $= \frac{1}{3} \times (5^2 + 3^2 + 5 \times 3) \times .7854 \times 12 = 153.982$ cubic inches. The content of the given frustum is $= 13 \times 282 = 3666$ cubic inches. Because the frustums have the same altitude, they are to one another in the same ratio as the areas of their ends, or as the squares of the diameters of their ends. Hence $153.982 : 3666 :: 5^2 : 595.3687$, the square of the diameter of the greater end; the square root of which is 24.4002, the greater diameter: and as $5 : 3 :: 24.4002$ the greater diameter : 14.6401, the less diameter. L.

Third Solution.

Let the given frustum be completed into a cone. Then the part added will be a cone similar to the whole cone. Let D and d denote the diameters of the greater and less ends of the given frustum respectively; P and p the perpendiculars of the whole cone and the part added, the contents of these cones being denoted by c and c. Then by the question $5 : 3 :: D : d$, but by similar triangles $D : d :: P : p$; therefore $5 : 3 :: P : p$, and therefore $5 - 3 : 3 :: P - p$ (or 12 the height of the frustum) $: p$, the height of the part added $= 18$; hence $P = 18 + 12 = 30$ inches, the height of the whole cone. Because similar solids are to one another in the same ratio as the cubes of their altitudes, we have as $P^3 : p^3 :: c : c$, therefore $P^3 - p^3 : P^3 :: c - c$, (the given frustum) $: c$; that is, as $27000 - 5832$ or $21168 : 27000 :: 3666 : \dfrac{3666 \times 27000}{21168} = c = 4676.02$, the content of the whole cone. Hence, because $.7854 \times D^2 \times \frac{1}{3} P = c$,

$$D = \sqrt{\dfrac{c}{\frac{1}{3} \times .7854 \times P}} = \sqrt{\dfrac{4676.02}{\frac{1}{3} \times .7854 \times 30}} = 24.4002,$$ the

greater diameter. And $5 : 3 :: 24.4002 : 14.6401$, the less diameter. L.

The ingenious Mrs. Babara Sidway, in her answer to this question, proposed another very pretty

QUESTION.

If the frustum of the cone was to hold as much again : what would be the length of the part added to the greater end ?

First Solution to Mrs. Sidway's Question.

If $5a$ and $3a$ be the diameters of the first or given frustum, whose altitude is 12, and z the altitude of the frustum to be added : Then, by similar triangles. &c. $\frac{1}{6}az + 5a = $ the greater diameter of the part

added; and since the two contents are equal, we shall have $((\frac{1}{6}az+5a)^2 + (\frac{1}{6}az+5a) \times 5a+25a^2) \times z=(25aa+15aa+9aa) \times 12$, or $((\frac{1}{6}z+5)^2 + (\frac{1}{6}z+5) \times 5 + 25) \times z=49 \times 12 = 588$, or $z^3 + 90 z^2 + 2700z = 21168$. Hence $z = 6\cdot3847619 =$ the height of the part to be added.   H.

Second Solution to Mrs. Sidway's Question.

Let the given frustum be completed. Let the height and content of the whole cone be found as in the last solution to the original question: The height is 30 inches, and the content of the whole cone is 4676·02 cubic inches. To this add 3666 cubic inches, the content of the given frustum; the sum is 8342·02 cubic inches, which is the content of the cone, including the lengthened frustum. Then since similar solids are to one another as the cubes of their altitudes, we have as $4676\cdot02 : 8342\cdot02 :: 30^3 : \dfrac{8342\cdot02}{4676\cdot02} \times 30^3$, the cube of the altitude of the last mentioned cone; the cube root is $30 \times \sqrt[3]{\dfrac{8342\cdot02}{4676\cdot02}} = 36\cdot382$ nearly, from which taking 30, (the altitude of the cone, including the given frustum), there remains 6·382 for the length of the frustum to be added to the greater end of the given frustum, so that it may hold twice the quantity as at first.   L.

II.   QUESTION 18, by Mrs. Lydia Fisher.

What two numbers are those whose sum is $16\frac{1}{2}$, and the greater divided by the less gives the quotient 27.

Answer.   The numbers are $15\dfrac{51}{56}$ and $\dfrac{33}{56}$.

First Solution.

Put z for the less number: then, by the question, either $16\frac{1}{2} - z$ or $27z$ will express the greater: hence $27z = 16\frac{1}{2} - z$, or $28z = \dfrac{33}{2}$; then $z = \dfrac{33}{56} =$ the less number, and consequently $16\frac{1}{2} - \dfrac{33}{56} = 15\frac{51}{56} =$ the greater.   H.

Second Solution.

Let $x =$ the less number; then $27x =$ the greater, and $28x = 16\frac{1}{2} = \dfrac{33}{2}$; hence the rest will be as above.   H.

Third Solution.

Since, as above, the numbers are in proportion as 27 to 1, and their sum $= 16\frac{1}{2}$: hence as $28 : 16\frac{1}{2} :: \begin{cases} 1 : \dfrac{33}{56} = \text{the less number.} \\ 27 : 15\frac{51}{56} = \text{the greater.} \end{cases}$ H.

Fourth Solution.

Let x be the greater number, y the less : $s = 16\frac{1}{2}$ and $q = 27$.

Then by the question $x + y = s$ and $\dfrac{x}{y} = q$.

From the first $x = s - y$ and from the second $x = qy$.

Therefore $qy = s - y$ ∴ $qy + y = s$ ∴ $y = s \div (q+1) : x = qs \div (q+1)$.

And thus the values of x and y in the equations $x^n \pm y^n = s$,

and $\dfrac{x}{y} = q$ are found; viz. $x = q \sqrt[n]{\dfrac{s}{q^n \pm 1}}$, and $y = \sqrt[n]{\dfrac{s}{q^n \pm 1}}$.　L.

III.　QUESTION 19.

Find two fractions whose sum is $\dfrac{1}{7}$, and product $\dfrac{4}{847}$.

Answered by J. R. S.

The fractions $\dfrac{1}{11}$ and $\dfrac{4}{77}$.

First Solution.

Let $\dfrac{1}{14} + z$ and $\dfrac{1}{14} - z$ express the two numbers; then $\left(\dfrac{1}{14}+z\right) \times$

$\left(\dfrac{1}{14} - z\right) = \dfrac{1}{196} - zz = \dfrac{4}{847}$; hence $z = \sqrt{\left(\dfrac{1}{196} - \dfrac{4}{847}\right)} = \dfrac{3}{154}$;

consequently $\dfrac{1}{14} + \dfrac{3}{154} = \dfrac{1}{11} =$ the greater number, and $\dfrac{1}{14} -$

$\dfrac{3}{154} = \dfrac{4}{77} =$ the less number.　H.

Second Solution.

Put x and y for the greater and less numbers, $s = \frac{1}{7}$ and $p = \frac{4}{847}$.

Then by the question $x + y = s$, and $xy = p$.

The square of the first equation is $x^2 + 2xy + y^2 = s^2$.

　　　　　Four times the second is　$4xy = 4p$.

Take the lower from the upper, $x^2 - 2xy + y^2 = s^2 - 4p$.

The square roots give　　　$x - y = \sqrt{(s^2 - 4p)}$.

But by the question　　　$x + y = s$.

Half the sum and half the difference of these equations, give

$$x = \frac{s + \sqrt{(s^2 - 4p)}}{2},$$

$$y = \frac{s - \sqrt{(s^2 - 4p)}}{2}.$$

In this way the values of x and y may be easily found from the equations $x^n \pm y^n = s$ and $xy = p$; and which values are

$$x = \sqrt[n]{\frac{s + \sqrt{(s^2 \mp 4p^n)}}{2}}, \text{ and } y = \sqrt[n]{\frac{s \infty \sqrt{(s^2 \pm 4p^n)}}{2}}.　\text{L.}$$

IV. QUESTION 20, *by Mr.* William Hawney.

A General who had served the King successfully in his wars, asked as a reward for his services, a farthing for every different file of ten men each, which he could make with a body of 100 men. The King thinking the request a very moderate one readily assented. Pray what sum would it amount to?

Answered by Mrs. Barbara Sidway.

$$\frac{91}{1} \times \frac{99}{2} \times \frac{93}{3} \times \frac{94}{4} \times \frac{95}{5} \times \frac{96}{6} \times \frac{97}{7} \times \frac{98}{8} \times \frac{99}{9} \times \frac{100}{10} =$$
17310309456440 farthings $=$ 180315723501. 9s. 2d.

Solution.

The number of variations of 10 in 100 are $\frac{100}{10} \times \frac{99}{9} \times \frac{98}{8} \times$
$\frac{97}{7} \times \frac{96}{6} \times \frac{95}{5} \times \frac{94}{4} \times \frac{93}{3} \times \frac{92}{2} \times \frac{91}{1} = $ 17310309456440 $=$ the number of farthings $=$ 180315723501. 9s. 2d. $=$ the sum demanded. H.

The following Theorem will answer all questions of this nature:

THEOREM. The number of combinations of *m* given things, all different from each other, taken by a given number *n* at a time, is equal to

$$\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \&c. \text{ to } n \text{ terms. } L.$$

V. QUESTION 21, *by Mr.* Gideon Cosier.

A gentleman riding by a common where a number of shepherdesses were tending their flock, after saluting them he asked the number of their sheep, when one of them answered, that if the flock was to be equally divided amongst the shepherdesses, the share of each would be twice as many sheep as there were shepherdesses: but if to one he gave one sheep, to another two, to the third four, &c. doubling the number to each succeeding shepherdess, then the last would have for her share as many sheep as there were in all. How many sheep were there?

Answered by Mr. Alexander Weedon, *and Mrs.* Barbara Sidway.

The number of sheep was 128.

Solution.

If z be put to denote the number of maids; then, by the question, $z \times 2z = 2zz$ will denote the number of the sheep, and $2zz = 2^{z-1}$; hence, multiplying by 2, we have $4zz = 2^z$; and, by extracting the

root, $2z = 2^{\frac{1}{2}z}$; From this equation, by trials, we easily find $z=8=$ the number of maids ; and then $2zz = 128$ the number of sheep.

When the value of z is an integer number, it is best found by trials as above. But if it be some infinite decimal, put the above equation into logarithms, and we have $\frac{1}{2}z \times \log. 2 = \log. 2 + \log. z$, or $1\cdot50515z - z = \cdot301030$, putting $z = \log.$ of z; then find the value of z by double position or the method of trial-and-error.   H.

<p style="text-align:center">VI.   QUESTION 22, <i>by Mr.</i> Amos Fish.</p>

A ship sails north, from a certain port; after running some days in this direction she alters her course and steers due west till her difference of longitude was a degree more than the difference of latitude made upon the first course. And it was also observed that a perpendicular from the point where the ship altered her course, to the right line joining the ship and the port sailed from, was 150 miles. Required the distance run in all, and the direct distance from the ship to the port sailed from.

<p style="text-align:center"><i>Answered by Mrs.</i> Babara Sidway.</p>

Sailed North......................188·256 miles.
Sailed West ....248·256
Distance run......................436·512
From the port sailed from to the ship's place   311·555

<p style="text-align:center"><i>Solution.</i></p>

From the right angle B let fall the perpendicular BD : If AB represent the distance sailed north, and BC the distance west; then will AC represent the distance between the place sailed from and the place arrived at, and BD 150 miles.
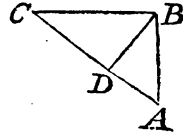
Now by right-angled triangles we have
$$\text{AC}^2 = \text{CB}^2 + \text{BA}^2,$$
and $2\text{AC} \times \text{BD} = 2\text{CB} \times \text{BA} ;$
hence, subtracting the second equation from the first, we have $\text{AC}^2 - 2\text{AC} \times \text{BD} = \text{CB}^2 - 2\text{CB} \times \text{BA} + \text{AB}^2 = (\text{CB} - \text{BA})^2$, and $\text{AC} = \text{BD} + \sqrt{(\text{BD}^2 + (\text{CB} - \text{BA})^2)} = 150 + \sqrt{(150^2 + 60^2)} = 150 + 30\sqrt{29} = 311\cdot554944214 =$ the distance from the place sailed from.

And, by adding the second equation to the first, we have $\text{AC}^2 + 2\text{AC} \times \text{BD} = \text{CB}^2 + 2\text{CB} \times \text{BA} + \text{AB}^2 = (\text{CB} + \text{BA})^2$, and $\text{CB} + \text{BA} = \sqrt{((\text{AC} + 2\text{BD}) \times \text{AC})} = \sqrt{(611\cdot5549 \&c. \times 311\cdot5549 \&c.)} = 436\cdot50082 \&c.) =$ the whole distance run.

Also, by adding and subtracting the difference and sum, we have CB and $\text{BA} = \dfrac{436\cdot50082 \pm 60}{2} = 248\cdot25041$ and $188\cdot25041$.

The *Construction** is evident from the above process, *viz.*

Make $AC = (BD + \sqrt{(BD^2 + (CB - BA)^2)}) =$ the sum of BD and the hypothenuse of a right-angled triangle whose two legs are BD and CB — BA : then the sum of the legs (CB + BA) being equal to $(\sqrt{((AC + 2BD)} \times AC))$ the mean proportional between AC and AC + 2BD, the legs CB, BA themselves will be = the half sum and difference of CB — BA and the said mean proportional.   H.

Another solution to this question, on the principles of Mercator's sailing.

Let $a =$ BD $= 150$; $b =$ the latitude, in minutes, of A the place sailed from; $m = 3437\frac{1}{4}$ the reciprocal of the length of an arc of 1 min. to the radius 1 ; $c = m \times : b + \frac{1}{6}b^3 + \frac{1}{24}b^5 + \frac{61}{5040} b7 + \&c. =$ the meridional parts in the latitude b, or c might be taken from a table of merid. parts ; $d = 60$, the difference between the difference of latitude and longitude, also put $x =$ the latitude of c, the place arrived at.

Then $m \times : x + \frac{1}{6} x^3 + \frac{1}{24} x^5 + \frac{61}{5040} x^7 + \&c. =$ the merid. parts in the lat. x.

Hence $x - b =$ AB, the proper difference of latitude ; and $- c + m \times : x + \frac{1}{6} x^3 + \frac{1}{34} x^5 + \frac{61}{5040} x^7 =$ merid. difference of latitude.

Now AD $= \sqrt{((x - b)^2 - a^2)}$: DB :: (AB : BC ::) merid. diff. of latitude : difference of longitude $= - \dfrac{ac}{\sqrt{((x - b)^2 - a^2)}} +$

$\dfrac{ma}{\sqrt{((x - b)^2 - a^2)}} \times : x + \frac{1}{6}x^3 + \frac{1}{24} x^5 \&c.$ But the difference of longitude is $=$ AB $+ 60 = x - b + d$. Then this value being equated to the other, there will result an equation from which the value of x may be found, and thence every thing else.   H.

THE PRIZE QUESTION FOR 1711.

In a level garden there are two lofty firs having their tops ornamented with gilt balls, one is 100 feet high, the other 80, and they are 120 feet distant at the bottom: now the owner wants to place a fountain in a right line between the trees, to be equally distant from the top of each ; and to make a walk or path, from the fountain, in every point of which he shall be equally distant from each of the balls : also at the end of the walk he would fix a pleasure house, so that a couch placed therein should be at the same distance from each ball, as the two balls are from each other. How must this be done ?

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* Another construction is given at question 252.   L.

Answered by Mr. Henry Beighton, *Mrs.* Anna Wright, *and Mr.* Ford, *Salt-Officer of Middlewich, and several others.*

From ball to ball ..·..........................121·65525
From each ball to the pleasure house .... .121·65525
From the lower-tree bottom to the house...... 91·65151
From the taller-tree bottom to the house ... ... 69·28203
From the house to the fountain..............·.... 52·67827
From the lower-tree bottom to the fount...... 75
From the higher-tree bottom to the fount...... 45

Solution.

Let AD be the one tree, and BC the other. Draw CD the distance of the two balls. On the middle of which and perpendicular to it draw GE, and E will be the fountain. For then DE will be = EC by *Eucl.* I. 4.

Also the required path, at the end of which stands the pleasure house, will be a right line EH perpendicular to AB, and = one leg of a right-angled triangle whose other leg is EA, and its hypothenuse AH = the base of another right-angled triangle ADH whose perpendicular is AD and hypothenuse DH = DC.—For, because of the path's continual equal distances from the balls, it will be the intersection of the horizontal plane and another plane perpendicular both to the line DC and the vertical plane ABCD), and this, intersection is perpendicular to AB by *Eucl.* XI. 19.
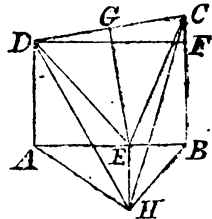
The Calculation. Since $(ED^2=) DA^2 + AE^2 = (EC^2 =) CB^2 + BE^2$; therefore, by equal subtraction $BC^2 - DA^2 = AE^2 - EB^2 = AB \times (AE - EB)$; and hence $AE - EB = \dfrac{BC^2 - DA^2}{AB} = \dfrac{100^2 - 80^2}{120} = 30$:

Consequently AE and $EB = \dfrac{AB \pm 30}{2} = \dfrac{120 \pm 30}{2} = 75$ and 45, the distances of the fountain from the bottoms of the two trees, And $DE = EC = \sqrt{(CB^2 + BE^2)} = \sqrt{(100^2 + 45^2)} = 5\sqrt{481} = 109·65856099$ the distance of each ball from the fountain. Also $DH = HC = CD = \sqrt{(DF^2 + FC^2)} = \sqrt{(120^2 + 20^2)} = 20\sqrt{37} = 121·6552506$. Hence $BH = \sqrt{(HC^2 - CB^2)} = \sqrt{(20^2 \times 37 - 100^2)} = 20\sqrt{12} = 69·2820323 =$ the distance of the pleasure house from the foot of the taller tree. $AH = \sqrt{(HD^2 - DA^2)} = \sqrt{(20^2 \times 37 - 80^2)} = 20\sqrt{21} = 91·651513899 =$ the distance of the same from the lower tree. And $EH = \sqrt{(HB^2 - BE^2)} = \sqrt{(20^2 \times 12 - 45^2)} = 5\sqrt{111} = 52·67826876 =$ the length of the walk, or the distance of the pleasure house from the fountain.   H.

Questions proposed in 1712, and answered in 1713.

I. QUESTION 23, by Mr. Lewis Evan.

A person at his death left his wife with child, and bequeathed his property, which was £7000, as follows, viz. if it was a son, the mother was to have one-third, and the son the rest of the property; but if a daughter, then the mother was to have two-thirds, and the daughter the rest. Now it happened that she had twins, a son and a daughter. This being a case not provided for in the will, the executors are perplexed how to divide the property equitably between the mother and the children. Required the share of each.

Answered by Mrs. Anna Wright.

The daughter's share is £1000, the mother's £2000, and the son's £4000.

Solution.

By the will, the mother's share was to be the double of the daughter's, and the son's double of the mother's; wherefore they will be in proportion as the numbers 1, 2, and 4; and the sum of these numbers being 7, it appears that the shares will be $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{4}{7}$ of the whole estate: viz.

$\frac{1}{7}$ of 7000l. $=$ 1000l. $=$ the daughter's share.
$\frac{2}{7}$ of 7000l. $=$ 2000l. $=$ the mother's share.
$\frac{4}{7}$ of 7000l. $=$ 4000l. $=$ the son's share. H.

Algebraic Solution.

Let x represent the daughter's share; then $2x$ will represent the mother's share, and $4x$ the son's.

Then by the question $x + 2x + 4x$ or $7x = 7000$: Hence $x = 1000$.

Therefore the three shares are 1000, 2000, and 4000 as before. L.

II. QUESTION 24, by Mr. Amos Fish.

A merchant had Holland, Muslin, Scotch linen, and Cambric, in all 30,000 ells, and he remarked, that if the number of ells of Holland was multiplied by those of Cambric, the product would be equal to that of the number of ells of Muslin multiplied by those of Scotch linen: And the sum of the squares of the number of ells of Holland and Cambric is 160,000,000. He also remarked that the number of ells of Holland was much greater than any of the others. How many ells were there of each?

Answered by Mr. John Newbold.

Holland 12000, Muslin 8000, Scotch cloth 6000, and Cambric 4000.

Solution.

This question is unlimited, for there are four unknown quantities, and only three conditions. And, to determine the rest, one of them must be assumed.

Put z, y, x, v for the four quantities; then

$z + y + x + v = 30,000 = a,$

$zv = yx,$

$zz + vv = 160,000,000 = b.$

To and from the third adding and subtracting the double of the first, we have $zz \pm 2zv + vv = b \pm 2xy$; hence $z \pm v = \sqrt{(b \pm 2xy)}$, then by substitution in the first, we get $y + x + \sqrt{(b + 2xy)} = a$; from this is obtained $y = a - \sqrt{(b + 2ax - xx)}$: by assuming $x = 6000$, this expression gives $y = 8000$. And then z and $v =$

$$\frac{\sqrt{(b + 2xy)} \pm \sqrt{(b - 2xy)}}{2} = 12000 \text{ and } 4000. \quad \text{H.}$$

III. QUESTION 25, *by Mr.* Alexander Weeden.

There is a piece of timber in the form of the frustum of a square pyramid, its length is 12 feet, the side of the square of the greater end is 21 inches, and that of the less end 3 inches, what is the content in cubic feet; and supposing the timber divided into pieces of a cubic foot each, by cuts parallel to the ends, what is the thickness of each, beginning at the greater end?

Answered by Francius.

The first solid foot's length is 4·0136, the second is 4·2208, the 3d is 4·4552, the 4th is 4·7272, the 5th is 5·0448, the 6th is 5·42, the 7th is 5·876, the 8th is 6·4408, the 9th is 7·1656, the 10th is 8·1352, the 11th is 9·5208, the 12th is 11·7096, the 13th is 15·8776, the 14th is 29·4824, and the 15th is 21·9104.

Solution.

By page 159 Mensuration, the solidity of the whole piece will be

$$\frac{21 \times 21 + 21 \times 3 + 3 \times 3}{3} \times 12 \times 12 = 24624 \text{ inches} = 14\frac{1}{4} \text{ feet,}$$

And if the frustum be supposed to be compleated to a pyramid; then, by similar triangles, as $21 - 3$ is to 21, or as 18 to 21, or as 6 to 7, so is 12 to 14 feet its altitude; and hence $\frac{21 \times 21}{3} \times 14 \times 12 = 24696$

inches $= 14·291\frac{2}{3}$ feet $=$ the content of the whole pyramid. Whence, as similar solids are as the cubes of their like dimensions, we shall

have $\sqrt[3]{14\cdot2916} : \sqrt[3]{13\cdot2916} :: 14$ feet or 168 inches $: 168\sqrt[3]{\frac{13\cdot2916}{14\cdot2916}}$

inches, and $168 - 168\sqrt[3]{\frac{13\cdot2916}{14\cdot2916}} = \frac{\sqrt[3]{14\cdot2916} - \sqrt[3]{13\cdot2916}}{\sqrt[3]{14\cdot2916}} \times 168 =$

$(\sqrt[3]{14\cdot2916} - \sqrt[3]{13\cdot2916}) \times 69\cdot228 =$ the altitude of the lowest or 1st foot.   And in like manner all the others will be found as below.

$: \sqrt[3]{14\cdot2916} - \sqrt[3]{13\cdot2916} : \times 69\cdot228 = 4\cdot0134 =$ alt. of 1st foot

$: \sqrt[3]{13\cdot2916} - \sqrt[3]{12\cdot2916} : \times 69\cdot228 = 4\cdot2203 = \dots$ 2d

$: \sqrt[3]{12\cdot2916} - \sqrt[3]{11\cdot2916} : \times 69\cdot228 = 4\cdot4557 = \dots$ 3d

$: \sqrt[3]{11\cdot2916} - \sqrt[3]{10\cdot2916} : \times 69\cdot228 = 4\cdot7273 = \dots$ 4th

$: \sqrt[3]{10\cdot2916} - \sqrt[3]{9\cdot2916} : \times 69\cdot228 = 5\cdot0442 = \dots$ 5th

$: \sqrt[3]{9\cdot2916} - \sqrt[3]{8\cdot2916} : \times 69\cdot228 = 5\cdot4205 = \dots$ 6th

$: \sqrt[3]{8\cdot2916} - \sqrt[3]{7\cdot2916} : \times 69\cdot228 = 5\cdot8765 = \dots$ 7th

$: \sqrt[3]{7\cdot2916} - \sqrt[3]{6\cdot2916} : \times 69\cdot228 = 6\cdot4406 = \dots$ 8th

$: \sqrt[3]{6\cdot2916} - \sqrt[3]{5\cdot2916} : \times 69\cdot228 = 7\cdot1649 = \dots$ 9th

$: \sqrt[3]{5\cdot2916} - \sqrt[3]{4\cdot2916} : \times 69\cdot228 = 8\cdot1354 = \dots$ 10th

$: \sqrt[3]{4\cdot2916} - \sqrt[3]{3\cdot2916} : \times 69\cdot228 = 9\cdot5210 = \dots$ 11th

$: \sqrt[3]{3\cdot2916} - \sqrt[3]{2\cdot2916} : \times 69\cdot228 = 11\cdot7093 = \dots$ 12th

$: \sqrt[3]{2\cdot2916} - \sqrt[3]{1\cdot2916} : \times 69\cdot228 = 15\cdot8778 = \dots$ 13th

$: \sqrt[3]{1\cdot2916} - \sqrt[3]{0\cdot2916} : \times 69\cdot228 = 29\cdot4827 = \dots$ 14th

the sum 122·0896 being taken from
12 feet or 144·0000 inches, we have

$21\cdot9104 =$ alt. of the $\frac{1}{4}$f. H.

IV. QUESTION 26, *by Mr.* John Burnet.

The depth of a standard bushel measure is 8 inches, and its diameter $18\frac{1}{2}$ inches : what must be the diameter of another bushel of the same capacity whose depth is $7\frac{1}{2}$ inches ?

Answered by Mr. Thomas Shepheard.

The diameter is 19·1067 inches.

Solution.

Since equal cylinders have their altitudes reciprocally proportional to the squares of their diameters, we shall have, As $\sqrt{7\frac{1}{2}} : \sqrt{8} ::$

B 4

$18\frac{1}{2} : 18\frac{1}{2} \sqrt{\frac{8}{7\frac{1}{2}}} = 18\frac{1}{2} \sqrt{\frac{16}{15}} = \frac{74\sqrt{15}}{15} = 19\cdot10671784$ inches $=$
the diameter required. H.

V. QUESTION 27, *by Mr.* John Wilson.

A person being asked his age replied, if to the months of my age you add their one half and one eighth, the sum will be 442: What was his age?

Answered by Mrs. Mary Nelson.

Twenty-two years and 8 months.

Solution.

This question being to find a number which being added to its one-half and one-eighth shall make 442; or, since the sum of 1, $\frac{1}{2}$, and $\frac{1}{8}$ is $\frac{13}{8}$, to find a number which multiplied by $\frac{13}{8}$ shall produce 442; wherefore as $13 : 8 :: 442 : \frac{442 \times 8}{13} = 34 \times 8 = 272$ months $= 22$ years 8 months $=$ his age. H.

Algebraic Solution.

Let $x =$ his age: then by the question $x + \frac{x}{2} + \frac{x}{8} = 442$, or $8x + 4x + x = 3536$. Hence $13x = 3536$, and $x = 272$ months $= 22$ years and 8 months. L.

VI. QUESTION 28, *by Mr.* Henry Beighton.

A person hired a horse in London at the rate of 3*d.* a mile, and proceeded to Bristol, lying west from London 94 miles; from Bristol he went to West Chester, lying due north, and from Chester he returned towards London till he arrived at Coventry, 66 miles from Chester. From Coventry he went again to Bristol, where having dispatched his affairs, he returned to Coventry and went thence directly to London. Now he is at a loss how to settle for the hire of the horse, not knowing the distances of Bristol and Chester, Coventry and Bristol, and Coventry and London. However he recollected that at Coventry the direct road to Bristol was at right angles to the London road. From the preceding data it is required to find the sum due?

This question is in effect the same as the following: In a right angled triangle, there are given one of the legs and the segment of

the hypothenuse, adjacent to the other leg, made by a perpendicular from the right angle to determine the triangle.

Then the distance which the traveller rode will be equal to the sum of the perimeter of this triangle, and twice the perpendicular from the right angle upon the hypothenuse.

Answered by Mrs. Mary Nelson.

£5 13s. 2¼d. is the sum due for the hire of the horse.

First Solution.

CONSTRUCTION. Make AB, AC perpendicular to each other, the former = 94 the distance between London and Bristol, and the latter = 66 the distance between Chester and Coventry. Bisect AC in D, and with centre D and radius DB describe a circle meeting AC produced in E and F; then draw CG perpendicular to AE, and with the hypothenuse AE construct the right-angled triangle AGE, by describing a semicircle on AE to meet CG in G; so shall E represent London, G Bristol, A Chester, and C Coventry.
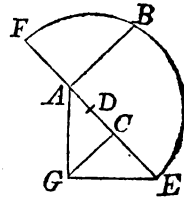
For by similar triangles, $GE^2 = EA \times EC =$ (by the construction) $EA \times AF = AB^2$, or $GE = AB$.

CALCULATION. The radius $ED = DB = \sqrt{(BA^2 + AD^2)} = \sqrt{(94^2 + 33^2)} = \sqrt{9925} = 5\sqrt{397} = 99.62429$. Hence $CE = ED - DC = 66.62429 =$ the distance from Coventry to London; $GA = \sqrt{(AE^2 - EG^2)} = \sqrt{((AD + DE)^2 - EG^2)} = \sqrt{(226.62429 \times 38.62429)} = 93.55855$; and $GC = \dfrac{EG \times GA}{AE}$ or $= \sqrt{(GE^2 - EC^2)} = 66.31141$. Then $GA + AE + EG + 2GC = 452.80566$ miles = the whole distance travelled; which at 3d a mile will produce 5l. 13s. 2d. 1.66792q.   H.

Second Solution.

If E represent London, G Bristol, A Chester, and C Coventry; then in the right-angled triangle AGE we have given the side GE, and the segment AC of the hypothenuse made by the perpendicular from the right-angle to construct it. Make GE of the given length and draw AGH perpendicular to GE making GH = half the given segment AC; with the centre H and radius HG describe a circle, and draw EH to meet it in D and E; make EA = EF, and the triangle will be constructed.
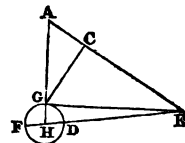
For GE being a tangent to the circle, $EG^2 = ED \cdot EF$ (36. III.), also $EG^2 = CE \cdot EA$ (8. VI.); therefore $ED \cdot EF = CE \cdot EA$: but $EA = EF$ by construction; therefore CE is $= ED$, and consequently $AC = DF = 2HG =$ the given segment.

CALCULATION. HE $= \sqrt{}$(GE^2 + GH^2) $= 99\cdot62429$, HE — HD $=$ DE $=$ EC $= 66\cdot62429$ the distance from Coventry to London ; and the rest as above. L.

PRIZE QUESTION, *by Mr.* Peter Hingeston, *junior, a scholar in Ipswich Grammar School.*

A dodecaedron is a solid composed of twelve regular pentagonal pyramids, whose vertices meet in the centre of the circumscribing sphere, and the bases of the pyramids form the superficies of the dodecaedron. Now suppose a dodecaedron having the side of each pentagon composing the superficies thereof 8 inches, and supposing every two of its composing pyramids to be hollowed out in the form of the greatest hemisphere, cylinder, cube, cone, triangular pyramid, and square pyramid: What will the remainder of the dodecaedron weigh after having been hollowed or scooped out as above described, supposing each cubic foot of the matter of which it is composed to weigh 60lb ?

Answered by Mrs. Barbara Sidway *and Mr.* Beighton.

The dodecaedron weighs, after all the twelve cavities are cut out, 60·6527 pounds.

Solution.

Let ABEFGH be one of the twelve pyramids constituting the dodecaedron, and c the centre of its base: draw CD perpendicular to BE, and the rest of the lines as in the figure.

Putting A $= 8 =$ BE one side of the pentagon or base, by page 410 Mensuration, we have AC$=$A$\sqrt{\dfrac{25 + 11\sqrt{5}}{40}} = 1\cdot113516$A ;

and CD $= \frac{1}{4}$A $\times$ tang. 54° $= \cdot68819095$A $=$ radius of the base of the cone. And, since the greatest inscribed cylinder is known to be $\frac{4}{9}$ of the cone, we shall have $\frac{13}{9}$ CD^2 $\times$ 3·14159 &c. $\times \frac{1}{3}$CA $= 2\cdot14916$A^2 $\times \frac{1}{3}$CA $=$ the cone and cylinder together.——But the greatest triangle in the pentagon is BFG, and the greatest quadrangle is BFGH; the sum of these two is the $\triangle$ GHB $+ 2 \triangle$ BFG. Now the $\triangle$ GHB $=$ GH $\times$ HB $\times \frac{1}{2}$s. $\angle$ H $= \frac{1}{2}$A^2 $\times$ s. 108° $= \cdot47552825$A^2, and $2 \triangle$ BFG $=$ BI $\times$ GF $=$ A^2 $\times \frac{1}{2}$ tang. $\angle$ BFI $=$ A^2 $\times \frac{1}{2}$ tang. 72° $= 1\cdot53884175$A^2; the sum of these two is 2·01437A^2 $=$ the bases of the triangular and quadrangular pyramids together; and consequently their contents together will be 2·01437A^2 $\times \frac{1}{3}$AC.—— Adding this to the sum of the cone and cylinder above found, we have 4·16353A^2 $\times \frac{1}{3}$AC $= 1\cdot545386$A^3 $=$ the sum of these four solids.
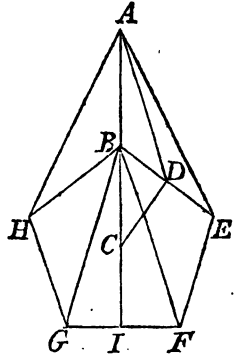
Again, $DA = \sqrt{(AC^2 + CD^2)} = \dfrac{5 + \sqrt{5}}{4}A = 1.309071A$, and by si-

milar triangles $AD : DC :: CA : \dfrac{AC \times CD}{DA} = \dfrac{5 + 3\sqrt{5}}{2G}A = .5854102A$

$=$ the radius of the inscribed hemisphere; and consequently its so-
lidity is $(.5854102A)^3 \times \frac{2}{3} \times 3.14159$, &c. $= .4201837A^3$.

Also, a square may be inscribed in a pentagon by placing one of its
sides parallel to a side of the pentagon; and, by calculation, it appears
that if z be the side of the square, $z \div 1.0604974$ will be the side of
the pentagon; hence, if this square be one face of the cube, and the
pentagon the section of the pentagonal pyramid parallel to the base
and distant from it the height of the cube, we shall have, by si-
milar figures, $AC : BE :: AC - z : z \div 1.0604974$; hence $z =$

$\dfrac{1.0604974AC}{1.0604974A + AC} \times A = .5431802A =$ the side of the cube: And con-

sequently the cube of this, or $.1602626A^3 =$ the content of it.

Collecting now these three sums together, and multiplying by 2, we
have $4.2516646A^3$ for the contents of the twelve cavities.

But, by p. 408 Mensuration, $5A^3 \sqrt{\dfrac{47 + 21\sqrt{5}}{40}} = 7.6631189A^3$

$=$ the content of the dodecaedron.

Consequently the difference of these two $= 3.4114543A^3 =$
1746.6646016 inches $= 1.0108013$ feet $=$ solidity remaining; which
at 60lb. a foot, weighs 60.648077lb.

N. B. *False numbers (but no method of solution) are given to this
question in* The Diarian Repository.

*A false solution is also given to question 25 in that book, dissimilar
solids being used as similar ones.* H.

———◈◈✕◉◌———

Questions proposed in 1713 *and answered in* 1714.

I.   QUESTION 29, *by Mrs*. Sarah Brown.

A lady being warmly importuned by her lover to consent to their
union, answered, that she thought herself then too young: the lover
desired to be informed when she would think herself old enough to
make him happy; she replied when the square of her age dimi-
nished by $\frac{1}{9}$ and then increased by $\frac{1}{3}$ of the same was 891 : What
would then be her age?

Answered by Mr. Lingen, *from Ireland*.

Twenty-seven years.

Solution.

Since $\frac{1}{9}$ taken from 1 leaves $\frac{8}{9}$, and this increased by $\frac{1}{3}$ or $\frac{3}{9}$ becomes $\frac{11}{9}$; hence the question is to find a number whose square multiplied by $\frac{11}{9}$ may produce 891.

Consequently, 11 : 9 :: 891 : the square of the number,

Or, $\sqrt{11} : \sqrt{9} :: \sqrt{891} : \sqrt{\dfrac{891 \times 9}{11}} = \sqrt{(81 \times 9)} =$

27 = the number sought.   н.

Algebraical Solution.

Let $x =$ her age: then by the question $xx - \dfrac{xx}{9} + \dfrac{xx}{3} =$

891; reduced $xx = 729$; therefore $x = 27$.   L.

II.   QUESTION 30, by A. W.

In London, on the morning of the 10th of May, happening to hold my walking-cane upright upon level ground, the sun shining brightly, I observed that the lengths of the cane and its shadow were exactly equal: What was the time of the day when this happened?

Answered by Mrs. Adway.

Thirteen minutes and sixteen seconds past nine o'clock.

Solution.

Supposing the sun's declination May 10th, 1712, to be 20° 16' 32"; we shall have given the three sides of a spherical triangle, to find an angle: viz. the complement of the declination = 69° 43' 28", the complement of the latitude = 38° 28', and the complement of the altitude = 45°, to find the hour angle, or angle between the two former; and which by calculation comes out 41° 41' 4" = in time to 2h. 46m. 44s. which being taken from 12h. we have 9h. 13m. 16s. for the time required.   н.

III.   QUESTION 31, by Mr. Rob. Wilson.

Suppose a ladder 100 feet long, placed against a perpendicular wall 100 feet high, how far would the top of the ladder move down the wall, by pulling out the bottom thereof 10 feet?

Answered by Mr. John Boswell and Mr. T. Busey.

·50126 feet = 6·01512 inches.

Solution.

By right-angled triangles, $\sqrt{(100^2 - 10^2)} = 30\sqrt{11} = 99\cdot4987437$

which taken from 100, there remains ·5012563 feet = 6·0150756 inches = the number required.   H.

IV.   QUESTION 32, *by Mr.* Lover.

Three men began talking about their money; the first said to the others, two-thirds of your money would make mine 86 nobles; and the second said to the others, three-fifths of your money would make mine 86 nobles; but if the other two lent five-eighths of their money to the third it would make his money amount to 86 nobles. What sum did each possess?

Answered by Mr. Edens.

The first had $30\frac{8}{37}$ nobles = $10l.$ $1s.$ $5\frac{1}{4}d$. . . . . 187.
Second $44\frac{6}{37}$ = $14l.$ $14s.$ $4\frac{1}{4}d$. . . . . . . .888.
Third $39\frac{17}{37}$ = $13l.$ $3s.$ $5d.$ . . . . .329.

Solution.

Putting $a = 86$ nobles, and z, y, x for the 1st, 2d, and 3d person's nobles respectively; we shall have, by the question,

$z + \frac{2}{3}y + \frac{2}{3}x = a$, Or $3z + 2y + 2x = 3a$,
$y + \frac{2}{3}z + \frac{3}{5}x = a$, Or $3z + 5y + 3x = 5a$,
$x + \frac{5}{8}z + \frac{5}{8}y = a$, Or $5z + 5y + 8x = 8a$.

Taking the 1st equation from the 2d, we have $3y + x = 2a$, or $x = 2a - 3y$; this substituted in the 1st and 3d, we have

$$\left. \begin{array}{l} 4y - 3z = a \\ 19y - 5z = 8a \end{array} \right\} \; ; \text{ from 3 times the latter of these equations}$$

subtract 5 times the former, and there will be obtained $37y = 19a$;

hence $y = \frac{19a}{37} = 44\frac{6}{37}$ nobles = $14l.$ $14s.$ $4d.$ $3\frac{13}{37}q$. Then $x =$

$2a - 3y = \frac{17a}{37} = 39\frac{17}{37}$ nobles = $13l.$ $3s.$ $5\frac{3}{37}d$. And $z = \frac{4y - a}{3} =$

$\frac{15a}{37} = 30\frac{8}{37}$ nobles = $10l.$ $1s.$ $5d.$ $1\frac{7}{37}q$.   H.

General Solution.

1st.   Let there be given the two equations

$$ax + by = c$$
$$a'x + b'y = c'$$

to find the values of x and y.

Multiplying the first by a' and the second by a, they become

$$aa'x + a'by = ca'$$
$$aa'x + ab'y = c'a ;$$

and subtracting the first from the second, we have

$$(ab' - a'b)y = c'a - ca',$$
$$\text{or } y = \frac{c'a - ca'}{ab' - a'b}.$$

In like manner, if the first equation be multiplied by b', and the second by b, and the latter product subtracted from the first, we have

$$(ab' - a'b)\, x = cb' - c'b$$

$$\text{or } x = \frac{cb' - c'b}{ab' - a'b}.$$

In this case, where there are only two unknown quantities, it is evident what factors will render the co-efficients of x and y, in the two equations equal to one another, as above; but when there are more unknown quantities than two, it is not so obvious what factors will answer; therefore in what follows I shall make use of a method for determining them, which is applicable to all cases, whatever may be the number of equations.

In the preceding example if we multiply by the indeterminate factors m and n, the equations become

$$amx + bmy = cm$$

$$a'nx + b'ny = c'n$$

and subtracting the second equation from the first, we have

$$(am - a'n)x + (bm - b'n)y = cm - c'n.$$

Now in order that y may disappear from this equation, $bm - b'n$ must be $= 0$, then $x = \frac{cm - c'n}{am - a'n}.$

But $bm - b'n$ will be $= 0$, when $m = b'$ and $n = b$; and these values being substituted for m and n in the expression for x, we have

$$x = \frac{cb' - c'b}{ab' - a'b}.$$

Again, to make x disappear from the equation, $am - a'n$ must be $= 0$, then $y = \frac{cm - c'n}{bm - b'n}.$ But $am - a'n = 0$, when $m = a'$ and $n = a$, therefore substituting for m and n their values, we have $y = \frac{ca' - c'a}{ba' - b'a}$; or changing the signs in the numerator and denominator,

$$y = \frac{ac' - a'c}{ab' - a'b}.$$

When one of the unknown quantities has been determined by this method, the other may be found, without repeating the process, by only changing the coefficients; or writing a in place of b, and a' in place of b', or the contrary. Thus, if in the value for x we write a for b, and a' for b', it becomes $\frac{ca' - c'a}{ba' - b'a}$, the value of y. And if in the value for y we write b for a and b' for a', we have $\frac{cb' - c'b}{ab' - a'b}$ for the value of x.

2nd. Let there be three equations

$$ax + by + cz = d$$

$$a'x + b'y + c'z = d'$$

$$a''x + b''y + c''z = d''$$

to find the values of x, y, and z.

Multiplying by the indeterminate factors m, n, and p, we have

$$amx + bmy + cmz = dm$$
$$a'nx + b'ny + c'nz = d'n$$
$$a''px + b''py + c''pz = d''p,$$

and subtracting the last from the sum of the other two, there remains

$(am + a'n - a''p)x + (bm + b'n - b''p)y + (cm + c'n - c''p)z = dm + d'n - d''p.$

Now in order that y and z may disappear from this equation, and x only remain,

$$bm + b'n - b''p = 0, \text{ or } bm + b'n = b''p$$
$$cm + c'n - c''p = 0, \text{ or } cm + c'n = c''p;$$

then $x = \dfrac{dm + d'n - d''p}{am + a'n - a''p}.$

From the equations $bm + b'n = b''p$

and $cm + c'n = c''p,$

we have, by the first case, $m = \dfrac{p(c'b'' - b'c'')}{bc' - b'c}$, and $n = \dfrac{p(bc'' - b''c)}{bc' - b'c}$;

and since p is arbitrary we may take it equal to the denominator $bc' - b'c$, then $m = c'b'' - b'c''$, and $n = bc'' - b''c$. These values being substituted for m, n, and p, in the expression for x, we have

$$x = \frac{d(c'b'' - b'c'') + d'(bc'' - b''c) - d''(bc' - b'c)}{a(c'b'' - b'c'') + a'(bc'' - b''c) - a''(bc' - b'c)}.$$

In like manner, if we make

$$am + a'n = a''p$$
$$\text{and } cm + c'n = c''p,$$

x and z will disappear from the equation, and we shall have

$y = \dfrac{dm + d'n - d''p}{bm + b'n - b''p}.$ And, proceeding as before, we find $m = c'a'' - a'c''$, $n = ac'' - ca''$, and $p = ac' - ca'$; therefore

$$y = \frac{d(c'a' - a'c'') + d'(ac'' - ca'') - d''(ac' - ca')}{a(c'a'' - a'c'') + a''(ac'' - ca'') - a''(ac' - ca')}.$$

Lastly, if we make $am + a'n = a''p$

and $bm + b'n = b''p$

x and y will disappear from the equation, and we shall have

$z = \dfrac{dm + d'n - d''p}{cm + c'n - c''p}.$ And substituting for m, n, and p their values found as above, namely, $m = b'a'' - a'b''$, $n = ab'' - ba''$, and $p = ab' - ba'$, we have

$$z = \frac{d(b'a'' - a'b'') + d'(ab'' - ba'') - d''(ab' - ba')}{c(b'a'' - a'b'') + c'(ab'' - ba'') - c'(ab' - ba')}.$$

These values being multiplied out, and the terms so arranged as to be positive and negative alternately, we have

$$x = \frac{db'c' - dc'b'' + cd'b' - bd'c' + bc'd'' - cb'd''}{ab'c' - ac'b'' + ca'b'' - ba'c' + bc'a'' - cb'a''}$$

$$y = \frac{ad'c'' - ac'd'' + ca'd'' - da'c'' + dc'a'' - cd'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}$$

$$z = \frac{ab'd'' - ad'b'' + da'b'' - ba'd'' + bd'a'' - db'a''}{ac'b'' - ab'c'' + ca'b'' - ba'c'' + bc'a'' - cb'a''},$$

the signs in the first and third expressions being changed in order that the denominator of each may be the same.

When one of the unknown quantities is determined, the others may be found without going through the separate operations, by only changing the coefficients; thus, if in the first value of x, we substitute the respective coefficients of y in place of those of x, viz. b in place of a, b' in place of a', and b'' in place of a'', we shall have the value of y: and if we substitute the coefficients of z instead of those of x, viz. c in place of a, c' in place of a', and c'' in place of a'', we shall have the value of z.

The reason of this will be obvious if we suppose the unknown quantities y and z to change places successively with x in the original equations.

To apply these formulæ to the example in question, where

$$2x + 2y + 3z = 3a$$
$$3x + 5y + 3z = 5a$$
$$8x + 5y + 5z = 8a$$

we have $a = 2$, $a' = 3$, $a'' = 8$, $b = 2$, $b' = 5$, $b'' = 5$, $c = 3$, $c' = 3$, $c'' = 5$, $d = 3a$, $d' = 5a$, and $d'' = 8a$.

And substituting these values in the above expressions,

$$x = \frac{3 \cdot 5 \cdot 5 - 3 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 5 - 2 \cdot 5 \cdot 5 + 2 \cdot 3 \cdot 8 - 3 \cdot 5 \cdot 8}{2 \cdot 5 \cdot 5 - 2 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 5 - 2 \cdot 3 \cdot 5 + 2 \cdot 3 \cdot 8 - 3 \cdot 5 \cdot 8} \times a = \frac{-17}{-37} \times a = \frac{17a}{37}$$

$$y = \frac{2 \cdot 5 \cdot 5 - 2 \cdot 3 \cdot 8 + 3 \cdot 3 \cdot 8 - 3 \cdot 3 \cdot 5 + 3 \cdot 3 \cdot 8 - 5 \cdot 5 \cdot 8}{2 \cdot 5 \cdot 5 - 2 \cdot 3 \cdot 5 + 3 \cdot 3 \cdot 5 - 2 \cdot 3 \cdot 5 + 2 \cdot 3 \cdot 8 - 3 \cdot 5 \cdot 8} \times a = \frac{-19}{-37} \times a = \frac{19a}{37}$$

$$z = \frac{2 \cdot 5 \cdot 8 - 2 \cdot 5 \cdot 5 + 3 \cdot 3 \cdot 5 - 2 \cdot 3 \cdot 8 + 2 \cdot 5 \cdot 8 - 3 \cdot 5 \cdot 8}{2 \cdot 5 \cdot 5 - 2 \cdot 3 \cdot 5 + 3 \cdot 3 \cdot 5 - 2 \cdot 3 \cdot 5 + 2 \cdot 3 \cdot 8 - 3 \cdot 5 \cdot 8} \times a = \frac{-13}{-37} \times a = \frac{13a}{37},$$

as before.

This method may be extended with the same facility to as many unknown quantities as we please; thus, if there be four equations

$$ax + by + cz + dw = e$$
$$a'x + b'y + c'z + d'w = e'$$
$$a''x + b''y + c''z + d''w = e''$$
$$a'''x + b'''y + c'''z + d'''w = e''';$$

then multiplying the first by m, the second by n, the third by p, the fourth by q, and subtracting the last product from the sum of the other three, we find $(am + a'n + a''p - a'''q) x + (bm + b'n + b''p - b'''q) y + (cm + c'n + c''p - c'''q) z + (dm + d'n + d''p - d'''q) w = em + e'n + e''p - e'''q.$

Therefore to find x, we have

$$bm + b'n + b''p - b'''q = 0$$
$$cm + c'n + c''p - c'''q = 0$$
$$dm + d'n + d''p - d'''q = 0$$

and $x = \dfrac{em + e'n + e''p - e'''q}{am + a'n + a''p - a'''q}.$

The values of m, n, and p must now be found from the formulæ in case 2nd. and substituted in the last expression, we shall then have

the value of x; and from thence by changing the coefficients we shall obtain the values of y, z, and w. The values of x, y, z, and w are as below.

The numerator of x is

$$eb'c''d''' - eb'd''c''' + ed'b''c''' - ed'c''b''' + ec'd''b''' - ec'b''d'''$$
$$+ bd'c''e''' - bd'e''c''' + bc'e''d''' - bc'd''e''' + be'd''c''' - be'c''d'''$$
$$+ ce'b''d''' - ce'd''b''' + cd'e''b''' - cd'b''e''' + cb'd''e''' - cb'e''d'''$$
$$+ db'e''c''' - db'c''e''' + dc'b''e''' - dc'e''b''' + de'c''b''' - de'b''c'''.$$

The numerator of y is

$$ae'c''d''' - ae'd''c''' + ad'e''c''' - ad'c''e''' + ac'e''d''' - ac'd''e'''$$
$$+ ed'c''a''' - ed'a''c''' + ec'a''d''' - ec'd''a''' + ea'd''c''' - ea'c''d'''$$
$$+ ca'e''d''' - ca'd''e''' + cd'a''e''' - cd'e''a''' + ce'd''a''' - ce'a''d'''$$
$$+ de'a''c''' - de'c''a''' + dc'e''a''' - dc'a''e''' + da'c''e''' - da'e''c'''.$$

The numerator of z is

$$ab'e''d''' - ab'd''e''' + ad'b''e''' - ad'e''b''' + ae'd''b''' - ae'b''d'''$$
$$+ bd'e''a''' - bd'a''e''' + be'a''d''' - be'd''a''' + ba'd''e''' - ba'e''d'''$$
$$+ ea'b''d''' - ea'd''b''' + ed'a''b''' - ed'b''a''' + eb'd''a''' - eb'a''d'''$$
$$+ db'a''e''' - db'e''a''' + de'b''a''' - de'a''b''' + da'e''b''' - da'b''e'''.$$

The numerator of w is

$$ab'c''e''' - ab'e''c''' + ae'b''c''' - ae'c''b''' + ac'e''b''' - ac'b''e'''$$
$$+ be'c''a''' - be'a''c''' + bc'a''e''' - bc'e''a''' + ba'e''c''' - ba'c''e'''$$
$$+ ca'b''e''' - ca'e''b''' + ce'a''b''' - ce'b''a''' + cb'e''a''' - cb'a''e'''$$
$$+ eb'a''c''' - eb'c''a''' + ec'b''a''' - ec'a''b''' + ea'c''b''' - ea'b''c'''.$$

And the denominator of each is

$$ab'c''d''' + ab'd''c''' + ad'b''c''' + ad'c''b''' + ac'd''b''' - ac'b''d'''$$
$$+ bd'c''a''' - bd'a''c''' + bc'a''d''' - bc'd''a''' + ba'd''c''' - ba'c''d'''$$
$$+ ca'b''d''' - ca'd''b''' + cd'a''b''' - cd'b''a''' + cb'd''a''' - cb'a''d'''$$
$$+ db'a''c''' - db'c''a''' + dc'b''a''' - dc'a''b''' + da'c''b''' - da'b''c'''.$$

If there were 5 unknown quantities, the numerators and denominators of each would consist of 120 terms.   L.

v.   QUESTION 33, by Mr. Peter Walter.

In what time would a heavy body fall freely from a height of seven miles, to the surface of the earth, the force of gravity being supposed uniform, and no resistance from the air, also in what time would sound move to the same height, its velocity being uniform and 1142 feet per second.

Answered by Mr. John Newbold *and Mrs.* Mary Nelson.

Forty-eight seconds.

Solution.

Since sound flies 1142 feet in a second of time, we shall have 7

miles $\div$ 1142 feet $=$ 36960 feet $\div$ 1142 $=$ 32$\frac{103}{571}$ $=$ 32·36427 se-
conds $=$ the time of the ascent of the sound.

Again since the times of descent are as the roots of the spaces, and
16$\frac{1}{12}$ feet is the space fallen in the 1st second, wherefore $\sqrt{16\frac{1}{12}}$:
$\sqrt{36960}$ :: 1 second : $\sqrt{\dfrac{36960}{16\frac{1}{12}}}$ $=$ 47·93778 seconds $=$ the time of
the body's descent. H.

' Mrs. Sidway *also answered this question, and at the same time pro-
posed the following one.*

From what height must a heavy body fall so that the time of descent
to the earth's surface may be equal to the time in which sound would
move the same height; and with what velocity would the body strike
the ground?

Mrs. Sidway's *Question solved.*

Putting $a =$ 16$\frac{1}{12}$, $b =$ 1142, and $x =$ the height of the castle in
feet. Then, as in the last solution, $\dfrac{x}{b}$ will express the time of the
sound's ascent, and $\sqrt{\dfrac{x}{a}} =$ the time of the body's descent; hence $\dfrac{x}{b}$
$= \sqrt{\dfrac{x}{a}}$, and $x = \dfrac{bb}{a} =$ 81087$\frac{77}{573}$ feet $=$ 15 miles 1887$\frac{77}{573}$ feet $=$
the height of the castle.

And since the velocity is as the root of the space descended, and
2a the velocity acquired by falling through a, we shall have $\sqrt{a}$:
$\sqrt{x} :: 2a : 2\sqrt{ax} = 2\sqrt{bb} = 2b =$ double the velocity of sound, or
2284 feet per second. H.

VI. QUESTION 34, *by Mr.* Josiah Clayton.

Suppose an army consisting of 60,000 soldiers drawn up in the form
of an oblong or rectangle upon a vast plain; how many acres will
they cover, allowing 2$\frac{1}{4}$ yards from man to man; the sides of the rect-
angle having the proportion of 3 to 2?

Answered by Mr. J. Lewes *and Mr.* Jos. Boydall.

Ninety-two acres, 3 roods, 35$\frac{1}{4}$ perches.

Solution.

As the sides must be in proportion to each other as 2 to 3, and
their product 60,000, it is evident that they must be 200 and 300.
Then the ground upon which they stand will be $=$ 299 $\times$ 2$\frac{1}{4}$ $\times$ 199
$\times$ 2$\frac{1}{4}$ $=$ $\frac{7199621}{16}$ yards $=$ 92 acres, 3 roods, 35$\frac{1}{4}$ perches. H.

PRIZE QUESTION, *by Mrs.* Anna Wright.

What star of the second magnitude was nearest to the north pole at the creation, supposing it to have happened 5716 years since; also what was its distance from the pole at that time.

Answered by Mr. John Edens.

The third star in Draco, reckoned from the tip of the tail was the nearest to the north pole at the creation, and its distance from the pole was 6° 48′ as appears from the following calculation :

	°	′	″
The longitude of the first in ♈ 1712..........	29	11	14
The star in Draco from ♈ ....ad.	124	33	30
Longitude of that star, 1712..............	153	44	44
Moved in 5716 years ....sub.	80	52	53
Remains the long. at the creation..........	72	51	51

Lat. according to Tycho, 66° 36′ north. Then
As the cotang. 23° 30′ : is to the radius ::
So is the sine 72° 51′ : to the tang. 22° 33′, which subtract from the compl. of the star's lat. leaves 50′ 12″. Then

As the cosine   22° 33′   To the cosine   50′ 12″
So is the cosine 23 30   To the cosine   6° 48

Six degrees forty-eight minutes the distance of that star from the pole at the creation.

Additional Solution.

The annual precession of the stars being 50·336 seconds of a degree, the precession from the creation to the year 1760 will be $=$ $(5716 + 48) \times 50·336 = 5764 \times 50·336 = 290136$ seconds $=$ 80° 35′ 36″. To this precession add 3 signs or 90°, the longitude of the north pole of the equator, and we shall have 5s. 20° 35′ 36″ for the present longitude of the star required; and the star of the second magnitude, whose longitude is nearest to that, and latitude 66 nearly, was the pole star required; which appears to be the third star in Draco, reckoned from the tip of the tail, which is the star marked, by Bayer, *α*.

Now to find the distance of this star from the pole at the creation.
By the catalogue of stars at the end of the Nautical Almanac for the year 1773, the latitude of *α* Draconis is 65° 21′ 15½″, (its complement 23° 38′ 44½″), and its longitude in the year 1760 was 154° 2′ 46″; from this subtract (80° 35′ 36″) the precession since the creation, and we have 73° 27′ 10″ for the longitude of this star at the creation, whose complement is 16° 32′ 50″: Also, since the diminution of the inclination of the earth's axis is 47″ in 100 years, in 5764 years its change will be 45′ 9″, which being added to 23° 28′ 17″ the inclination in the year 1760, we shall have 24° 13′ 26″ for the angle made by the axes of the ecliptic and equator at the creation.

c 2

Hence in an oblique spherical triangle, we have given two sides and the included angle, to find the third side; *viz.* given one side = 24° 13′ 26″ the distance of the pole of the equator from that of the ecliptic, the other side = 23° 38′ 44½″ the distance of the star from the pole of the ecliptic, and the included ∠ = 16° 32′ 50″ the complement of the star's longitude: Then, by trigonometry, is found the 3d side = 6° 43′ 10″ the required distance of the star from the pole of the equator at the creation.   H.

———◦◦◦◦◦◦—

Questions proposed in 1714, and answered in 1715.

I.   QUESTION 35, *by Mr.* John Wilson.

A father informed his daughter's lover that the cube of the number of pounds in his estate was 352947 × 64 × 9; and that he designed to give three-sevenths of it as her marriage portion; and being an odd kind of fellow, refuses to give his consent to the match till the lover has discovered what the portion is. The young gentleman has puzzled himself a great deal to no purpose, and is almost distracted for fear of losing both the money and his mistress; pray set him at ease by informing him what the portion is ?

Answered by Mr. Peter Walter, *Mr.* James Cale, *of Portsmouth, and Mr.* William Vorley, *Writing-master, at Holbech, Lincolnshire.*

The father's estate, 588*l.* The daughter's portion, 252*l.*

Solution.

Since the cube of the number of pounds in the whole estate is equal to 352947 × 64 × 9 = 203297472, the number itself must be equal to the cube root of this number, which is 588*l.* = the whole estate. Consequently $\frac{3}{7}$ of 588 = 252*l.* is the lady's portion.   H.

II.   QUESTION 36, *by Mrs.* Barbara Sidway.

From a given cone to cut the greatest cylinder possible.

Answered by Mr. Beriff, *Mr.* Hall, *Mr.* Wylde, *and Mr.* Clayton.

The cone must be cut at *one-third* of its altitude.

Solution.

Let ACB be the given cone, DE the diameter, and PQ the altitude of the greatest inscribed cylinder; then the solidity of the cylinder is $= DE^2 \times \cdot 7854 \times PQ$, therefore $DE^2 \times PQ$ must be a maximum. But DE has to AP a given ratio, namely that which the diameter of the base has to the altitude, and therefore when $DE^2 \times PQ$ is the greatest, $AP^2 \times PQ$ will also be the greatest; and this is the case when AP is double of PQ (Theo. xxiv. page 44, vol. 3, of the course;) hence the cone must be cut at a third of its altitude from the base.   L.
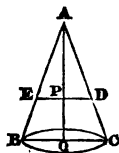
OTHERWISE, by FLUXIONS.

Put $AQ = a$, $BC = b$, and $x = AP$; then by similar triangles $a :$ $b :: x : \dfrac{b}{a}x = DE$; whence $DE^2 \times PQ = \dfrac{b^2}{a^2} \times x^2 \times (a - x)$;

therefore, leaving out the constant factor $\dfrac{b^2}{a^2}$, $x^2(a-x)$ must be a maximum.   Taking fluxions $2ax\dot{x} - 3x^2\dot{x} = 0$, and reducing, $x = \frac{2}{3}a$, whence $a - x = PQ = \frac{1}{3}a$, as before.   L.

III.   QUESTION 37, by Mr. John Hodge, of Truro, in Cornwall.

It was settled, by a party of nine persons, to defer a certain disagreeable business as many days as they could change places at dinner each day; how long would that be?

Answered by Mr. Thomas Dodd, *Mr.* Lingen, *and Mr.* Nath. Kew.

Nine hundred and ninety-three years, and 186 days.

Solution.

The time required will be $= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 = 362880$ days; which, at $365\frac{1}{4}$ days to the year, come to $993\frac{186}{365}$ years.   H.

The following theorem will answer all questions of this kind.

THEOREM.   The number of permutations, or changes that can be made of any given number of things m, all different from each other, is equal to $1 \times 2 \times 3 \times 4 \times 5 \times$ &c. to m terms.   L.

IV.   QUESTION 38, by Mr. Crabb, of Whitlackington, in Somerset.

An architect agreed to build an obelisk in the form of a cone, at 7d. per cubic foot, the height was to be to the circumference of its base as 5 to 4, and if the cone were cut by a plane parallel to the

base at the height of 10 feet, the content of the part intercepted be-
tween this plane and the vertex would be 1250 cubic feet. He more-
over agreed to build a chamber in the obelisk in the form of a cube,
the greatest that the cone would possibly admit of. What must be
the dimensions of the cone, as also of the chamber to be formed in
the same, and what will it come to at the price agreed upon ?

Answered by Mr. John Newbold, *Mr.* Lewis Evan, *Mr.* Abel Ragg,
    Mr. John Andrew, *Mr.* Nich. Stevens, *and Mrs.* Nelson.

	feet	inch.
The circumference of the pyramid at the base....	41	6
The whole height of the cone...................	51	10
The side of the room.........................	7	11

The content in solid feet when the room is deducted, 1878·1093.
The charge of the building, 5*l.* 15*s.* 6*d.*

Solution.

Put $a = \cdot785398$ &c. and $5x$ and $4x$ for the altitude and circum-
ference of the upper part whose content is 1250. Then its solidity is
$\dfrac{5x^3}{3a} = 1250$; hence $x = \sqrt[3]{750a} = 8\cdot382697$. Consequently the
altitude of the whole cone $= 5x + 10 = 51\cdot913485$ feet $= 51$ feet
10·96182 inches, and the circumference of its base $= 4x + 8 =$
41·53078 feet $= 41$ feet 6·369456 inches: Also, by similar solids,

$$(5x)^3 : (5x + 10)^3 :: \left(\frac{5x^3}{3a} = \right) \; 1250 : \left(\frac{x+2}{x}\right)^3 \times 1250 = 2375\cdot14$$

$=$ solidity of the whole cone.

Now, if $\text{A} = 5x + 10 = 51\cdot913485$ the altitude of the whole
cone, and $z =$ the side of the inscribed cubical room; by similar fi-
gures we shall have $\text{A} : \dfrac{\text{A}}{10a} :: \text{A} - z : \frac{1}{2}z\sqrt{2}$; hence $z =$

$\dfrac{\text{A}}{1 + 5a\sqrt{2}} = 7\cdot921364$ feet $= 7$ feet 11·056368 inches $=$ the side
of the room ; the cube of which gives 497·05 for the vacuity.

Taking this from the whole solidity (2375·14) we obtain 1878·09
for the solid part; which, at 7*d.* a foot, produces 5*l.* 15*s.* 6*d.*
2·52*q.* μ.

v.   QUESTION 39, *by Mr.* Richards, *of St. Thomas's, near Exon.*

The monument erected in Babylon by Queen Semiramis at her
husband Ninus's tomb, is said to have been one solid block of mar-
ble in the form of a pyramid ; the base was a square whose side was
20 feet and the height of the monument was 150 feet ; now suppose
this monument was sunk in the Euphrates, what weight would be

sufficient to raise the apex of it to the surface of the water, and what weight would raise the whole of it above the water?

Answered by Mr. Newbold.

	Ton.	cwt.	qr.	lb.
To raise it to the surface.............	939	3	2	10
To be added to raise it out..........	573	12	3	25
The weight in the air..............	1512	16	2	7

This calculus is according to Mr. *Ward's* tables of specific gravities, and differs considerably from those taken from the Philosophical Transactions.

Solution.

The solidity of the pyramid will be $20^2 \times \frac{150}{3} = 20000$ feet $=$ 34560000 cubic inches. Now by *Ward's* table of specific gravities, a cubic inch of marble weighs 1·568859 ounces avoirdupois, and an inch of salt water ·594894 ounces; and their difference is ·973965. Whence, the weight required to raise the point of it to the surface of the water will be ·973965 × 34560000 = 33660230·4oz. = 939 *tons* 3*cwt.* 2*qr.* 12*lb.* 6·4oz.

And it is evident that the whole weight of the marble will be necessary to raise the whole of it above the water.   H.

VI. QUESTION 40, *by Mr.* John Newbold.

A cooper is ordered to make a tub in the form of the frustum of a cone capable of holding 19·98 ale gallons; the internal diameter of the greater end is to be 30 inches, and this is also to be the diameter of a sphere which will circumscribe the vessel. What will be the diameter of the less end and depth of the tub?

Answer.

The least diameter 18·236 inches, and depth 11·9104 inches.

Solution.

Put $2a = 30 =$ the bottom diameter, $2x =$ the top diameter, and $a =$ the content in ale gallons. Then, by the nature of the circle, $\sqrt{(aa - xx)} =$ the perpendicular depth; and, by page 159 Mensuration, we have $4 \times (aa + ax + xx) \times \sqrt{(aa - xx)} \times \frac{·78539 \&c.}{3} = 282a$, where $282 =$ the inches in a gallon; or $(aa + ax + xx) \times \sqrt{(aa - xx)} = \frac{846a}{3·14159 \&c.}$. Now when $a = 19·98$ as in the question, there is no number, taken as the value of x, will produce so great a content, the number 19·98 being above the maximum.

But to find the greatest content the case will admit of, put the fluxion of $(aa + ax + xx)^2 \times (aa - xx) =$ to nothing, and we shall have $0 = (aa - xx) \times (a + 2x) - x \times (aa + ax + xx) = a^3 + a^2x - 2ax^2 - 3x^3$; or in numbers $x^3 + 10x^2 - 75x - 1125 = 0$; or, putting $z = \frac{1}{5}x$, $z^3 + 2z^2 - 3z + 9 = 0$; the root z of this equation is $= 1 \cdot 9394$; consequently $2x = 10z = 19 \cdot 394 =$ the value of the less diameter when the content is a maximum ; and then the depth $\sqrt{(aa - xx)}$ is $= 11 \cdot 444$, and the greatest content $=$

$$(30^2 + 30 \times 19 \cdot 349 + 19 \cdot 349^2) \times \frac{11 \cdot 444}{3} \times \frac{\cdot 78539 \&c.}{282} = 19 \cdot 739$$

ale gallons. H.

<center>PRIZE QUESTION.</center>

In gauging a spheroidical ale cask, I found the diameter of one head to measure 18·1 inches, that of the other 16, the bung diameter 20, and the distance between the two heads 20·6 inches, also, by the cask lying a little obliquely, I observed that the liquor just rose to, or touched the upper extremities of the two heads. Having noted these dimensions, I was informed that there were in the cask a ball of iron weighing 60*lb.* another ball of lead weighing 90*lb.* and a cube of box, a foot square. Pray what quantity of liquor was in the cask ?

Answered by Mr. John Newbold, *Mr.* John Edens, *Mrs.* Anne Morgan, *Mrs.* Mary Nelson, *of Newmarket, Mr.* William Crabb, *and Mr.* R. Sandford.

	inches.
The spheroid's greatest distance from bung to head....	12·053
The lesser distance................................	8·547
The content of the cask in ale gallons.............	20·763
The iron ball equal to cubic inches................	217·048
The leaden ball	219·717
The cask's vacuity.................................	117·814
The box emerged	1723·000
The sum, cubic inches	2277·579
which are equal to ale gallons	8·076
which deducted from the whole content leaves	12·687

ale gallons, the true quantity of liquor remaining in the cask.

Solution.
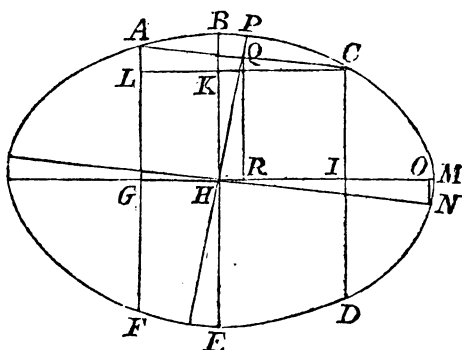
Let ACDF be the cask, BE the conjugate axe, and HM the semi-transverse; CL ∥ to HM, HN ∥ to AC, HQP the semi-conjugate of HN, and NO and QR ∥ BE.

By the nature of the ellipse, we have

$\sqrt{(HB^2 - GA^2)} : \sqrt{(HB^2 - IC^2)} :: LK : KC$; and, by compos. &c.

$$\sqrt{(HB^2 - GA^2)} + \sqrt{(HB^2 - IC^2)} : LC \; (= LK + KC)$$

$$\left\{ \begin{aligned} \sqrt{(HB^2 - GA^2)} : LK &= \frac{LC \times \sqrt{(HB^2 - GA^2)}}{\sqrt{(HB^2 - GA^2)} + \sqrt{(HB^2 - IC^2)}} \\ &= \frac{20 \cdot 6 \times \sqrt{(10^2 - 9 \cdot 05^2)}}{\sqrt{(10^2 - 9 \cdot 05^2)} + \sqrt{(10^2 - 8^2)}} = 8 \cdot 5463. \\ \sqrt{(HB^2 - IC^2)} : KC &= \frac{LC \times \sqrt{(HB^2 - IG^2)}}{\sqrt{(HB^2 - GA^2)} + \sqrt{(HB^2 - IC^2)}} \\ &= \frac{20 \cdot 6 \times \sqrt{(10^2 - 8^2)}}{\sqrt{(10^2 - 9 \cdot 05^2)} + \sqrt{(10^2 - 8^2)}} = 12 \cdot 0537. \end{aligned} \right.$$

And, $\sqrt{(EK \times KB)} : KC :: BH : HM = \dfrac{BH \times KC}{\sqrt{(EK \times KB)}} = \dfrac{120 \cdot 537}{6}$

$= 20 \cdot 0895.$

Also, by right-angled △ s, $AC = \sqrt{(CL^2 + LA^2)} = \sqrt{(20 \cdot 6^2 + 1 \cdot 05^2)}$

$= 20 \cdot 62674$; and sine of $\angle ACL = \dfrac{LA}{AC} = \cdot 0509048 = $ sine $\angle NHM$,

which put $= s$, and its cosine $= c$.

Then, by the nature of the ellipse,

$MH^2 : HB^2 :: MH^2 - HO^2 : ON^2$;

but $HO = HN \times c$, and $ON = NH \times s$, wherefore

$MH^2 : HB^2 :: MH^2 - HN^2 \times c^2 : HN^2 \times s^2$, or by subtraction

$MH^2 - HB^2 : HB^2 :: MH^2 - HN^2 \times (s^2 + c^2) : HN^2 \times s^2$,

   or since $s^2 + c^2 = 1$

$MH^2 - HB^2 : HB^2 :: MH^2 - HN^2 : HN^2 \times s^2$, or

$(MH^2 - HB^2) \times s^2 : HB^2 :: MH^2 - HN^2 : HN^2$,

   and by addition

$(MH^2 - HB^2) \times s^2 + HB^2 : HB^2 :: MH^2 : HN^2$, and hence

$$\text{NH} = \frac{\text{MH} \times \text{HB}}{\sqrt{(\text{HB}^2 + s^2 \times (\text{MH}^2 - \text{HB}^2))}} =$$

$$\frac{\text{MH} \times \text{HB}}{\sqrt{(\text{HB}^2 \times c^2 + \text{MH}^2 \times s^2)}} = 20{\cdot}01094.$$

Also, its semi-conjugate PH $= \sqrt{(\text{MH}^2 + \text{HB}^2 - \text{HN}^2)} = 10{\cdot}15629.$

But HR $= \dfrac{\text{IH} - \text{GH}}{2} = 1{\cdot}7537$; and RQ $= \dfrac{\text{GA} + \text{IC}}{2} = 6{\cdot}525$; and, by right-angled triangles, HQ $= \sqrt{(\text{QR}^2 + \text{RH}^2)} = 8{\cdot}703511$; consequently QP $=$ PH $-$ HQ $= 1{\cdot}45278.$

Then, by cor. 2, page 275 Mensuration, the content of the segment APC will be $3{\cdot}14159$ &c. $\times$ MH $\times$ HB^2 $\times$ QP^2 $\times \dfrac{\text{HP} - \frac{1}{3}\text{PQ}}{\text{PH}^3} =$ 122·9787 inches $=$ the vacuity.

But the content of the box cube is $12^3 = 1728$; all of which will be immersed, because it is heavier than ale. Also, using Ward's table of specific gravities, the solidity of the iron ball will be $\dfrac{60 \times 16}{4\cdot422979}$ $= 217{\cdot}0483$; and that of the leaden one $\dfrac{90 \times 16}{6\cdot553885} = 219{\cdot}717.$

The sum of these three added to the above segment, gives 2287·744 for the sum of the deductions.

Now the content of the whole cask is (2EB^2 + FA^2) $\times$ ·2618 HG + (2EB^2 + CD^2) $\times$ ·2618 HI $=$ (2EB^2 $\times$ GI + FA^2 $\times$ HG + CD^2 $\times$ HI) $\times$ ·2618 $= 5855{\cdot}3142.$

From this whole content then taking the above sum of the deductions, we obtain 3567·5702 inches; which divided by 282, we have 12·65096 for the number of gallons required. ц.

Questions proposed in 1715 and answered in 1716.

I. QUESTION 41, by Mr. Dod.

A person who had been tippling all day, was, upon his coming home at night, questioned as to what money he had spent; in order to evade a direct answer, he made the following reply: if the square of the number of pence which he had spent were multiplied by five times their own square root, the product would be equal to the pence in 162l. What sum had he spent?

Answered by Mr. Richard Sandford and Silvia.

If the pence in 162l. be b, and the pence spent be $= a$; then this theorem solves the question; $a = \sqrt[5]{(b^2 \div 25)} = 36$ pence.

Solution.

Let x denote the pence spent. Then, by the question $x^2 \times 5\sqrt{x}$

$= 162 \times 240 = 38880$, or $x^2 \sqrt{x} = 7776$; hence, by squaring, $x^4 \times x$ or $x^5 = 7776^2$; and consequently $x = 7776^{\frac{2}{5}} = 36d. = 3s.$ H.

II. QUESTION 42, *by Mr.* William Taylor.

A may-pole having been broken by a blast of wind, it was found that the broken part measured 63 yards, and by falling the top had made a hole or mark in the ground, at the distance of 30 yards from the foot of the pole; it is required to determine what was the height of the pole when standing upright.

Answered by Mr. Jos. Peacock.

The height of the may-pole was 118·3985 yards.

Solution.

It is evident, from the question, that the part broken off (63) is the hypothenuse of a right-angled triangle, whose perpendicular is the part left standing, and base equal to (30) the given distance which the top struck the ground from the pole; and consequently, by right-angled triangles, $\sqrt{(63^2 - 30^2)} = \sqrt{3069} = 55\cdot3985559 =$ the part left standing. To this adding 63 the part broken off, we obtain 118·3985559 for the whole length required. H.

III. QUESTION 43, *by Mrs.* Boydell.

An aged sire, with his two daughters, came
To town, and was desir'd to tell his name.
My name, says he, 's a thousand more than you,
And I. and if you add the other two,
The number of this present year they'll shew.
The * num'ral letters of our names (well set
With the first letter of the alphabet)
Make seventeen hundred, and fourteen complete.
I'll further add (to make the problem plain)
Each single name five letters doth contain.

Answered by Philo Tipperus, *and Mr.* P. Proson.

Whilst Lidia for Lydai you cunningly write,
   You think you may surely ensnare us;

But David is aged, and Lucia bright,
   So 'tis easy to —— Philo Tipperus

David	1006
Lidia	552
Lucia	156

The daughters who to town with David came,
Lucia was one, and Lidia's t'other's name.

P. Proson.	1714

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* D. *stands for* 500, C. *for* 100, L. *for* 50, V. *for* 5, I. *for* 1, &c.

IV. QUESTION 44, *by Mr.* T. Hayward.

Two men, one stronger than the other, have to remove a large stone, weighing 300lbs. with a bearing barrow, whose length is 6 feet: now the weakest cannot undertake to carry more than 100lbs. how must the stone be placed upon the barrow, so as just to allow him that weight for his share?

Answered by Silvia.

The centre of gravity of the stone must be laid on the barrow at two feet from the strongest man.

In questions of this nature the result will be the greater strength multiplied by the shorter length, is equal to the lesser strength multiplied by the greatest length; and here the barrow's weight is not accounted for, otherwise it would require some skill in staticks to resolve it.

V. QUESTION 45, *by Mr.* Thomas Shepheard.

Suppose a person whose height is 5 feet 7 inches, travels 10,000 miles in the arc of a great circle; how much will his head have gone farther than his feet, the circumference of the earth being 21600 miles?

Answered by Silvia.

It is evident in the case proposed, that while the feet desrcibe an arch on the earth's surface equal to 10,000 miles, the head will describe a similar arch of a circle, whose radius exceeds the semi-diameter of the earth 5 feet seven inches : Therefore as the semi-diameter of the earth, is to the semi-diameter increased by 5 feet 7 inches : So is 10,000 geographical miles, to an arch exceeding 10,000 miles by 16·23 feet, according to *Van Culen's* proportion of the diameter to the circumference, *viz.* as 1 to 3·141592653589793.

Additional Solution.

Since the earth's radius is $\dfrac{21600}{2 \times 3\cdot14159 \&c.}$ inches, and 5 feet 7 inches $= 67$ inches; by similar sectors, as radius : radius $+ 67 :: 10000$: 10000 + the excess travelled by the head; or, by division, as radius : 67 :: 10000 : the said excess; that is, as $\dfrac{21600}{2 \times 3\cdot14159 \&c.}$: 67 :: $10000 : \dfrac{10000 \times 67 \times 2 \times 3\cdot14159 \&c.}{21600} = \dfrac{6700 \times 3\cdot14159 \&c.}{108} =$ 194·8951 inches $=$ 16·2412 feet $=$ the excess required. H.

VI. QUESTION 46, *by Mr.* Ed. Elphick.

Suppose a smooth inclined plane from the top of a lofty tower meets the level ground at the distance of 304 feet from the bottom of the tower, and suppose a weight of 40lbs. will sustain a weight of 56lbs. when placed upon the inclined plane: What is the height of the tower?

Answered by Silvia.

By a known principle in mechanics, the accelerating velocity, or weight of bodies on an inclined plane, is to their accelerating velocities or weight in their perpendicular descent, as the sine of the angle of inclination, to the radius; or as the perpendicular to the length of the plane, considered as an. hypothenuse; and therefore in this the proportion of the perpendicular, and the length of the plane, being given as 40 to 56, or as 1 to 1·4.   Let the perpendicular sought be $= x$, the hypothenuse will be $1·4x$, and the base 304 feet $= a$, then $1·96x^2 — x^2 = 0·96x^2 = a^2$ by 47 Euclid I. and by division and evolution,

$$x = \left(\frac{aa}{0·96}\right)^{\frac{1}{2}} = 310·27 \text{ feet required.}$$

Additional Solution.

It is evident that this problem may be easily *Constructed* thus: Make a right-angled triangle whose hypothenuse and perpendicular are 56 and 40, or 7 and 5, or any two numbers proportional to these; then make another similar triangle whose base may be 304; and its perpendicular will be the height of the castle required.

And from the same principle, the *Calculation* may be given without algebra, thus: Since $\sqrt{(7^2 — 5^2)} = \sqrt{24} =$ the base of the first triangle to which the other is similar, we shall have as $\sqrt{24} : 5 ::$ $304 : \dfrac{304 \times 5}{\sqrt{24}} = \dfrac{304}{\sqrt{96}} = \dfrac{760}{\sqrt{6}} = 310·2687 =$ the castle's height. H.

PRIZE QUESTION.

Two ships A and B are supposed stationary in the channel between Dover (D), and Calais (C): an observer on board the ship A, finds the $\angle$ BAD $= 86°$, and the $\angle$ CAB $= 24°$: and an observer on board the ship B, finds the $\angle$ ABC $= 129°$, and the $\angle$ ABD $= 48°$.   How far are the two ships distant from each other, as also from Calais and Dover, supposing an object floating on the water to be driven from Calais to Dover in 2h. 25m. 10s. 10‴ 33⁗ 36‴‴ by the force of the waves 45 feet distant.

To this question, besides abundance of false answers, I received ten true ones, and because I see so few amongst so many ingenious correspondents, I am apt to believe it was as difficult, as it was an uncommon question; for which reason I shall here give you the solution, and the algebraic method by which the same is found.

Sir *Isaac Newton*, in his *Principia*, says, Let there be a pendulum, whose length from the point of suspension, to the centre of oscillation, is the breadth of any wave; then while the pendulum makes an oscillation, the wave will pass over a distance equal to its breadth. From which, and the nature of pendulums, is raised the following theorem. Let the distance run be $= a$, the length of a standard pendulum $= p = 39.2$ inches; the square of the vibrations in a minute $= f = 3600$, $s = 45$ feet or 540 inches, the breadth of the wave, and $t = 2$ hours 25 min. &c. $= 145.1696$ minutes, also $r = 45$ feet: then s

$: p :: f : \dfrac{p.f}{s}$; but $\sqrt{\dfrac{fp}{s}} = $ to the number of vibrations it will make in a minute, and multiplied by r will be the feet the wave will move in a minute; but $\dfrac{a}{r\sqrt{\dfrac{p.f}{s}}} = t$, and consequently, by reduction, a

$= rt\sqrt{\dfrac{p.f}{s}} = 105600$ feet $= 20$ miles sought, or the distance from Calais to Dover.

In this following figure c represents Calais, D Dover, A the first ship, B the second.

Then

From $\begin{cases} \text{Calais to Dover } c \text{ to } D \text{ 20 miles} \\ \text{Dover to 1st ship } B \text{ to } A \text{ 13·5192} \\ \text{Dover to 2d ship } D \text{ to } B \text{ 18·1583} \\ \text{Calais to 1st ship } c \text{ to } A \text{ 22·4008} \\ \text{Calais to 2d ship } c \text{ to } B \text{ 11·7238} \\ \text{1st ship to 2d ship } A \text{ to } B \text{ 13·0853} \end{cases}$
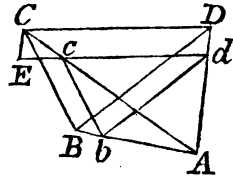
The angles DAB 86°, CAB 21°, DBA 48°, CBA 129°.

Additional Solution.

Perhaps the solution will appear a little clearer thus : Put $p = 39.2$ inches the length of the pendulum, $v = 60$ the number of vibrations it makes in a minute, $b = 45$ feet the breadth of a wave, and $t = 145.1696$ minutes the given time. Then, by the nature of pendulums,

$\sqrt{12b} : \sqrt{p} :: v : v\sqrt{\dfrac{p}{12b}} = $ the number of vibrations made in a minute by the pendulum whose length is the breadth of a wave, but for every vibration the wave will pass over a distance equal to its breadth b; hence the wave moves over $bv\sqrt{\dfrac{p}{12b}}$ or $v\sqrt{\dfrac{bp}{12}}$ feet in

each minute, which therefore drawn into t minutes makes $tv\sqrt{\dfrac{bp}{12}} = $

$tv\sqrt{\dfrac{45 \times 39\cdot 2}{12}} = tv\sqrt{(15 \times 9\cdot 8)} = 7tv\sqrt{3} = 145\cdot1696 \times 420\sqrt{3}$

$= 105605\cdot2$ feet $= 20$ miles nearly, $=$ the distance between Calais and Dover.

This distance being obtained, which is the side CD of a quadrilateral ABCD, of which the angles formed by the opposite side AB with the two diagonals BD, AC; as also with the other two sides AD, BC, being given, the figure will be *Constructed* (as in the British Oracle) thus:

At the extremities A, b, of any line Ab, make the angles bAC, bAD, bC, Abd equal to the given angles, and join c, d; so shall Abcd evidently be a quadrilateral similar to *that* formed by the two towns and two ships; and, consequently, if dc be produced to E, till dE be equal to 20 the given distance between Calais and Dover, and EC be drawn parallel to Ad and meeting AC in c, that then CB, CD being drawn parallel to cb, cd, and meeting Ab, Ad in B and D, the points A, B will be the two ships, c Calais, and D Dover.

The *Calculation* from this construction is also evident: *viz.* assuming A$b =$ any number, as suppose 10, in the two triangles Abc, Abd, we have given all the angles and the side Ab, to find the other sides; and hence those sides are AC $= 17\cdot11811$ miles, $bc = 8\cdot95914$ miles, A$d = 10\cdot33093$ miles, and $bd = 13\cdot86777$ miles. Then in the triangle bcd are given two sides with their included angle, to find the third side cd, which will be $= 15\cdot28859$ miles. Lastly, by similar triangles, as $cd = 15\cdot28859$: CD $= 20$

$$\begin{cases} Ab = 10 & : \text{AB} = 13\cdot03615, \\ \text{Ac} = 17\cdot11811 & : \text{AC} = 22\cdot39331, \\ Ad = 10\cdot33093 & : \text{AD} = 13\cdot51456, \\ bc = 8\cdot95914 & : \text{BC} = 11\cdot72004, \\ bd = 13\cdot86777 & : \text{BD} = 18\cdot74133. \quad \text{H.} \end{cases}$$

The latter part of the question resolved without assuming a similar figure.

ANALYSIS. Suppose the problem solved and that AB is the distance between the two ships as required. On CD let a segment of a circle DAC be described to contain an angle of 62° (CAD): this segment will evidently pass through the point A. Also on the the line CD let another segment CBD be described to contain an angle of 81° (CBD): this segment will evidently pass through the point B. Let the line AB when produced meet the circles in P and Q, and join CB, CQ, DP, DQ. Then the angle PCD is $=$ the given angle DBA $= 48°$, and the angle QDC is $=$ the given angle QAC $= 24°$. Hence this construction. Having described upon the line CD the segments DAC, CBD as in the analysis, draw CP to make the angle DCP $= 48°$, and DQ to make the angle CDQ $= 24°$; join QP meeting the circles in B and A. Then AB is the distance required.
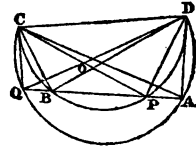
Calculation. In each of the triangles CQD, CDO, and CPD there are given one side CD and all the angles, to find the other sides : *viz.* CF, DQ, OD, and OC; also OP = CP — OC, OQ = QD — OD; hence in the triangle QOP, two sides and the included angle are given to find OPQ = CDB, and the angle OQP = ACD; then in the triangles CDA, DBC, one side and all the angles are given : whence AC, AD, BC, BD, and AB may be easily found. L.

———•••❂❍❂❍•••——

Questions proposed in 1716, *and answered in* 1717.

I.   QUESTION 47, *by Mr.* John Edens, *of Teneriffe, one of the Canary Islands.*

A triangular close whose sides are 20, 15, and 6 chains, is to be divided into two equal parts by a drain 3 feet wide. The drain is to be cut in such a manner as to remove the least quantity possible of the surface of the ground : how is the drain to be cut.

<center>*Answered.*</center>

AB is 6 chains. ⎱
CA 15 chains ⎬ per quest.
CB 20 chains ⎰
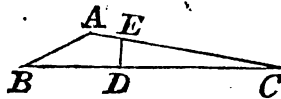
The content is easily found to be 2 acres, 3 roods, 17 perch. and 45280 parts of a perch. The drain will cut off an isosceles triangle, whose two legs will be equal, represented by EC, CD. By the doctrine of fluxions, these two theorems for finding CD and DE will come out,

$$CD = \sqrt{\frac{ab}{2}} = 269 \cdot 4435 \text{ yards, and}$$

$$DE = \sqrt{\frac{2ab + c^2 - a^2 - b^2}{2}} = 51 \cdot 59457 \text{ yards.}$$

Now 13837·97 yards the whole area — 51·59457 = 13786·37543 square yards after the drain is made.

<center>*Additional Solution.*</center>

By theorem 7 of *Simpson* on *The Maxima and Minima of Geometrical Quantities*, the required drain DE will cut off an isosceles triangle DCF; and since it must cut off half the whole (or the middle

of the drain, represented by DE, is supposed to cut off half the whole), and triangles which have one angle common being as the rectangles of their sides about the common angle, we shall have as

$$2 : 1 :: \text{AC} \times \text{CB} = 15 \times 20 : \frac{\text{AC} \times \text{CB}}{2} = 150 = \text{EC} \times \text{CD} = \text{EC}^2;$$

and hence $\text{EC} = \text{CD} = \sqrt{150} = 12{\cdot}2474487$ chains.

Now, by cor. 1, page 65 Mensuration, the triangle ABC is equal $\frac{1}{4}\sqrt{\left\{ ((\text{BC}+\text{CA})^2 - \text{AB}^2) \times (\text{AB}^2 - (\text{BC}-\text{CB})^2) \right\}} = \frac{1}{4}\sqrt{\left\{ (35^2 - 6^2) \times (6^2 - 5^2) \right\}} = \frac{1}{4}\sqrt{(41 \times 29 \times 11)} = 28{\cdot}590875$ square chains $=$ 13837·9835 square yards.

And, by the same, the triangle DCE is $= \frac{1}{4}\sqrt{\left\{ ((\text{DC}+\text{CE})^2 - \text{DE}^2) \times \text{DE}^2 \right\}}$ $= \frac{1}{4}\sqrt{((4\text{CD}^2 - \text{DE}^2) \times \text{DE}^2)} =$ (by writing $2\text{AC} \times \text{CB}$ for its equal 4CD^2) $\frac{1}{4}\sqrt{((2\text{AC} \times \text{CB} - \text{DE}^2) \times \text{DE}^2)}$.

But, by the question, this latter triangle is equal to half the former; wherefore $\text{DE}^2 \times (2\text{AC} \times \text{CB} - \text{DE}^2) = \dfrac{\text{AB}^2 - (\text{BC} - \text{CA})^2}{2} \times$

$\dfrac{(\text{BC} + \text{CA})^2 - \text{AB}^2}{2}$.

But $\dfrac{(\text{BC} + \text{CA})^2 - \text{AB}^2}{2} = 2\text{AC} \times \text{CB} - \dfrac{\text{AB}^2 - (\text{BC} - \text{AC})^2}{2}$; which being written for it in the last equation above, we have

$\text{DE}^2 \times (2\text{AC} \times \text{CB} - \text{DE}^2) = \dfrac{\text{AB}^2 - (\text{BC} - \text{CA})^2}{2} \times$

$\left(2\text{AC} \times \text{CB} - \dfrac{\text{AB}^2 - (\text{BC} - \text{AC})^2}{2} \right)$.

From this equation it is evident that DE^2 is $= \dfrac{\text{AB}^2 - (\text{BC} - \text{CA})^2}{2}$, and hence, as in the original solution, $\text{DE} = \sqrt{\dfrac{\text{AB}^2 - (\text{BC} - \text{CA})^2}{2}} =$

$\sqrt{\dfrac{6^2 - 5^2}{2}} = \sqrt{\dfrac{11}{2}} = \frac{1}{2}\sqrt{22} = 2{\cdot}34520788$ chains $= 51{\cdot}59457336$

yards $=$ the length of the drain, or $=$ its area since its breadth is one yard.——(*And this is a very curious and remarkable theorem.*)

By subtracting this from the whole area, we have 13786·389 square yards for the content remaining.   H.

<center>OTHERWISE.</center>

The above solution supposes the drain to be a parallelogram, and that the middle line of it, or the line bisecting its breadth, divides the whole field into two equal parts: but, strictly and mathematical, that is not the case; for, by the question, the parts on each side of the drain must be equal, and then the said middle line of the drain will divide the figure into unequal parts; as will thus appear.

Let DEFG represent the ditch, and HI the middle line. Through H draw *de* parallel to BC and meeting DG, EF in *d* and *e*: Then will be formed the two little equal isosceles triangles *d*HD, *e*HE, which will be both constant and given, because the angle C and the breadth of the ditch are so; and, since the parallelograms *e*I, HG are equal, as also the trapezium AF and triangle CDG by the question, it is evident that HI must be drawn cutting off the isosceles triangle CHI $=$ to $\frac{1}{2}$ the $\triangle$ ABC $-$ 2 the $\triangle$ *e*HE $=$ a given area; to do which needs no pointing out here.
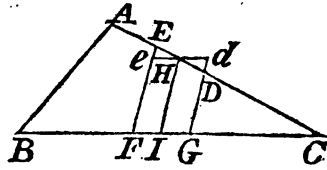
The calculation from hence would bring out the lines a little different from those above.   H.

The expressions in the original solution may also be obtained as follows :

Put AC $= a$, BC $= b$, AB $= c$, and $d =$ the angle DCE; then the triangle DCE being isosceles and equal to half ABC, AC $\times$ CB : DC $\times$ CE $:: 2 : 1$, or $ab : DC^2 :: 2 : 1$; hence $DC^2 = \dfrac{ab}{2}$, and DC $= \sqrt{\dfrac{ab}{2}}$.

Again, by formula 11, page 56, vol. 3, of the Course,

$DE^2 = DC^2 + CE^2 - 2DC \times CE \times \cos d = ab - ab \cos d,$

and $ab \cos d = \dfrac{a^2 + b^2 - c^2}{2}$; hence $DE^2 = ab + \dfrac{c^2 - a^2 - b^2}{2}$,

and $DE = \sqrt{\dfrac{2ab + c^2 - a^2 - b^2}{2}}$.   L.

II. QUESTION 48, *by Mr.* Massey.

A gentleman has his garden surrounded with walnut trees : upon gathering the fruit one season, he was informed that they had produced 2187 walnuts : upon every tree there were as many branches as there were trees in all, and upon each branch three walnuts, what was the number of trees ?

Answered by Mr. Dodd, *and Mr.* A. Naughley.

Let *a* be equal to the number of trees, then *aa* will be the number of all the branches ; and *aa* $\times$ 3 $=$ 2187 by the question. Therefore by division and evolution $a = \sqrt{(2187 \div 3)} = 27$ the number sought.

III. QUESTION 49, *by Mr.* Hawney.

Find four numbers in arithmetical progression whose common difference is 4, and their continual product 176985.

Answered by Mr. M. Moyle and Hesychia.

If the common difference be d, and the first number sought be equal to a, then $a \times (a + d) \times (a + 2d) \times (a + 3d) = a^4 + 6da^3 + 11d^2a^2 + 6d^3a = 176985$ per quest. Now by converging series, the value of (a) will be found $= 15$ just. Therefore $a + d = 19$, $a + 2d = 23$, and $a + 3d = 27$; but $15 \times 19 \times 23 \times 27 = 176985$ as required.

Additional Solution.

Let $d = 2$, half the common difference, and $p = 176985$. Then $x - 3d$, $x - d$, $x + d$, $x + 3d$ will denote the four numbers : the continual product of which is $x^4 - 10d^2 x + 9d^4 = p$, by the question. Add $16d^4$ to both sides, and we have $x^4 - 10d^2x^2 + 25d^4 = p + 16d^4$; the square roots give $x^2 - 5d^2 = \sqrt{(p + 16d^4)}$; hence $x^2 = 5d^2 + \sqrt{(p + 16d^4)}$ and $x = \sqrt{(5d^2 + \sqrt{(p + 16d^4)})} = 21$. Therefore the four numbers are $x - 3d = 15$, $x - d = 19$, $x + d = 23$ and $x + 3d = 27$.

The value of a, in the original solution may also be found by the resolution of a quadratic equation. Thus, the expression $a^4 + 6da^3 + 11d^2a^2 + 6d^3a$ being equal to $(a^2 + 3da)^2 + 2d^2 (a^2 + 3da)$, we shall have, putting p for the given product, $(a^2 + 3da)^2 + 2d^2 (a^2 + 3da) = p$. Hence, completing the square, &c. $a^2 + 3da = \sqrt{(p + d^4)} - d^2$; and completing the square again, we get $a = \sqrt{(1\frac{1}{4}d^2 + \sqrt{(d^4 + p)})} - 1\frac{1}{2}d = 15$, as before.   L.

IV.   QUESTION 50, by Philomathes.

What will a weight of two pounds troy lose by being carried to the height of seven miles above the earth's surface ?

Answered by Mr. Fr. Walker.

Forty-seven grains.

Solution.

Since the gravitation or weight of a body above the earth, is inversely as the square of its distance from the earth's centre; if the earth's radius be supposed equal to 3993 miles, we shall have $4000^2 : 3993^2 :: 2lb.$: the weight at 7 miles high; hence $4000^2 : 4000^2 - 3993^2 :: 2lb.$: the diminution,

Or $4000^2 : 7993 \times 7 :: 2lb. : \dfrac{7993 \times 7}{8000000} = \cdot 006993875 \, lb. = 36 \cdot 28472$ grains $=$ the weight lost, or the decrease of gravity at that height.   H.

v. QUESTION 51, *by Mr.* Tho. Fletcher.

Find three numbers, such, that if each is subtracted from the cube of their sum, the remainder shall be a cube.

Answered by Mr. Rob. Beales, *and Mr.* Beriff.

$\frac{494424}{2352637}$, $\frac{472696}{2352637}$, and $\frac{445000}{2352637}$: are three numbers which taken severally

from $\frac{512000}{2352637}$ the cube of their sum, leaves these 3 cubes, $\frac{17576}{2352637}$, $\frac{39304}{2352637}$,

and $\frac{64000}{2352637}$; whose roots are $\frac{26}{133}$, $\frac{34}{133}$, and $\frac{40}{133}$.

Additional Solutions.

This problem is the nineteenth question of the fifth book of Diophantus' Arithmetic; it occupied the attention of several of the greatest mathematicians at the close of the sixteenth and the early part of the seventeenth century, some of whose solutions as well as that of Diophantus which is the basis of all the rest, we shall exhibit after remarking that the publication of Xylander's Latin translation of Diophantus in 1575, forms an æra in the history of Algebra. For we find almost every great mathematician after that time a diligent cultivator of this elegant branch of it. Vieta, for instance, resolved the problems of finding two cubes whose sum shall be equal to the difference of two given cubes, or whose difference shall be equal to the sum of the two given cubes, or whose difference shall be equal to the difference of the given cubes : it is true his solutions bring out only one answer, but it is now well understood that by means of the first a second may be obtained, and so on : it is true also that his method seemed to lie under restrictions, but these too have been done away by the foregoing observation, and it remains that every solution must begin by his method even in the improved state of the science at this day. Bachet de Mezeriac, a very learned French gentleman, published the Greek text of Diophantus with a Latin translation, and large comments in 1621; and, by the help of the algebra of Vieta with which he every where seems well acquainted, threw new light upon this difficult author, whose text had not been much illuminated by Planudes and Xylander. But the remarks which Fermat, a gentleman of Tolouse, made in the margin of his Diophantus, and which were added to the new edition which appeared in 1670, seven years after the death of Fermat; contain the largest additions which have been made to the theory of numbers in modern times, at least previous to the labours of Euler and Lagrange and the theorem of Wilson. Lord Brounker and Dr. Wallis too, in England, distinguished themselves by solving the most difficult problems that Fermat and Frenicle were able to propose to them. Lord Brounker especially was the inventor of the method of continued fractions, and that of finding integral values of n such that $an^2 + 1$, a being a positive whole

number not a square, may be a square; a theorem of great import-
ance in this branch of mathematic science. Fermat and Wallis were
the first that resolved the sum of two given cubes into two other cubes,
but this problem, tho' thought a difficult one at that time, does not
now seem to have required any great stretch of thought after the three
problems resolved by Vieta. Descartes, a name ever memorable in
the hsitory of mathematics and philosophy, was, as appears by his
letters, well acquainted with the theory of rational numbers, or
what Gauss calls transcendental arithmetic : but Montucla is mis-
taken, in saying that he did not suffer the method by which he arrived
at his solutions to transpire, as there remains in the Exercitationes
Mathematicæ of Schooten, a solution of his at full length to a problem
of this kind, which was publicly proposed at Paris in 1633.   It oc-
cupies the twelfth section of the fifth book of that work.   Our pro-
blem too it seems was published as a challenge in 1610, at the end
of Ludolph van Ceulen's or Ludolphus à Collen's book on the cir-
cle.   But before we give his solution we shall set down that of Dio-
phantus.

Diophantus, Question 19, Book 5.

To find three numbers such, that being severally subtracted from
the cube of their sum, the remainders may be cube numbers.

For the sum of the three numbers put n, and for the numbers them-
selves put $\frac{7}{8}n^3$, $\frac{26}{27}n^3$, $\frac{63}{64}n^3$, it remains that the sum of the three
should be equal to n, therefore $\frac{4877}{1728}n^3 = n$, divide these equals by

n, then $\frac{4877}{1728}n^2 = 1$. But unity is a square. Therefore the coeffi-
cient of n^2 must be a square. And it arises from subtracting three
cubes, each less than unity, from the number three.   The problem is
therefore reduced to this : to find three cubes each less than unity, such
that their sum subtracted from the number three, may leave a remainder
that is a square number.   Now since each of the cubes must be less
than unity, if we suppose the sum of their roots less than unity,
they will each be much less than unity.   Thus the square which re-
mains will be greater than the number two.   Suppose that square
equal to $2\frac{1}{4}$. Therefore $\frac{3}{4}$ is to be divided into three cubes, or the
numerator of any fraction of equal value, with a cubic denominator.
Suppose $\dfrac{3 \times 6^3}{4 \times 6^3} = \dfrac{3 \times 216}{4 \times 6^3} = \dfrac{3 \times 54}{6^3} = \dfrac{162}{6^3}$.   Therefore   162

must be divided into three cubes.   Now 162 is the sum of 125 a
cube, and 37 the difference of the two cubes 64 and 27.   But in the
porisms it is shewn that the difference of every two cubes is equal to
the sum of two other cubes.   Let us recur to the original problem,
and take each of the cubes found, and having subtracted each of
them from unity, put the remainders for the numbers required and

the sum will be n. So that the remainder after the subtraction of any one of them will be a cube. It remains that the three together be equal to n. But the sum of the three is $2\frac{1}{4}n^3$. Therefore $n = 2\frac{1}{4}n^3$. Therefore $n = \frac{2}{3}$. Ad Positiones.

Scholium. Bachet has a long comment on this question in which he performs all the calculations that are only indicated by Diophantus, but owing to his method giving him only $\left(\frac{40}{91}\right)^3$ and $\left(\frac{303}{91}\right)^3$ for the cubes which together are equal to 37, and his neglecting to reduce his fractions to the least common denominator, his numbers are very high, when reduced by dividing their terms by 27 they are $\frac{68574961}{549353259}$, $\frac{162707336}{549353259}$, $\frac{134953109}{549353259}$ for the three numbers required in the question. But 37 is divisible into two cubes an infinity of ways, but perhaps not better for our present purpose than into $\left(\frac{19}{7}\right)^3 + \left(\frac{18}{7}\right)^3$. The numbers required in the question will then come out $\frac{69256}{250047}$, $\frac{31213}{250047}$, and $\frac{67229}{250047}$.

Bachet might have remarked that 162 is divisible not only into 125 and 37, but also into 64 a cube, and 98 the difference of the cubes 125 and 27, which would have furnished, in his own way, one other set of numbers.

We have only farther to remark, that the solutions to most problems of this kind depend upon a case being already known; and that Diophantus in fact reduces the problem to dividing 6 into three cubes, for $\frac{3}{4} = \frac{6}{8}$, the known case for which is $6 = 8 - 1 - 1$, from which an infinity of solutions may be obtained by putting $6 = (2 - bx)^3 + (cx - 1)^3 + (dx - 1)^3$ where $12b - 3(c + d) = 0$. If we take $c = 5$, and $d = 3$, and consequently $b = 2$,

$$x = \frac{56(c^2 + d^2) - 16cd}{21(c^3 + d^3) - cd\,(c + d)}$$ will come out $\frac{13}{24}$ and $6 = \left(\frac{11}{12}\right)^3 +$

$\left(\frac{41}{24}\right)^3 + \left(\frac{5}{6}\right)^3 = \left(\frac{22}{24}\right)^3 + \left(\frac{41}{24}\right)^3 + \left(\frac{15}{24}\right)^3$; if we take $c = 7, d = 5$,

b will be equal to 3, and $x = \frac{8}{21}$ and $6 = \left(\frac{6}{7}\right)^3 + \left(\frac{5}{3}\right)^3 + \left(\frac{19}{21}\right)^3 =$

$\left(\frac{88}{21}\right)^3 + \left(\frac{35}{21}\right)^3 + \left(\frac{19}{21}\right)^3$ and $162 = 27 \cdot 6 = \left(\frac{18}{7}\right)^3 + 5^3 + \left(\frac{19}{7}\right)^3$,

the numbers found above.

Diophantus seems to have chosen the least square that occurred between 2 and 3. Had he taken $2\frac{1}{4}$ the number instanced by Bachet, he would have had $\frac{9}{9} = \frac{6}{27}$ to divide into three cubes, which is just the same in effect as before, though Bachet does not seem to have perceived it. Another square $\frac{64}{25} = 2\frac{14}{25}$ subtracted from 3 leaves

the remainder $\frac{11}{25} = \frac{55}{125}$ to be divided into three cubes, and the known case is $55 = 64 - 8 - 1$. Again $3 - \frac{169}{81} = 3 - 2\frac{7}{81} = \frac{74}{81} = \frac{666}{729}$ and to divide 666 into three cubes, we have the evident case $666 = 729 - 64 + 1$. And from each of these positions for the square other sets of answers may be obtained infinite in number. It remains only to observe that the one solution to the question found out by Diophantus leads to infinite others. Thus let a^2 be a square, b^3, c^3, d^3 three cubes, such that $3 - a^2 = b^3 + c^3 + d^3$ by putting instead of a^2, $(a + fx)^2$, for b^3, c^3, d^3, $(b + gx)^3$, $(c + hx)^3$, $(d + kx)^3$, we shall have $-2afx - f^2x^2 = 3(b^2g + c^2h + d^2k)x + 3(bg^2 + ch^2 + dk^2)x^2 + (g^3 + h^3 + k^3)x^3$, whence by assuming $3(b^2g + c^2h + d^2k) + 2af = 0$, and dividing by x^4 we get $x = \dfrac{f^2 - 3(bg^2 + ch^2 + dk^2)}{g^3 + h^3 + k^3}$.

There is an observation of Fermat on this question of Diophantus, which very briefly indicates a solution, namely by putting $n - 1$ for the side of the square to be subtracted from the number three, and $1 - \frac{n}{3}$, $1 + n$, and $-pn$ for the sides of the three cubes; then rejecting superfluous terms and dividing by n^2 we have

$$-1 = \frac{1}{3} - \frac{n}{27} + 3 + n - p^3 n; \frac{27p^3 - 26}{27}n = \frac{13}{3} \therefore n = \frac{13 \cdot 9}{27p^3 - 26}.$$

In this solution n is confined to narrow limits and p must be taken accordingly: n must be greater than $1 + \sqrt{2}$ and less than $1 + \sqrt{3}$. And when n is so found, $(1 - \frac{n}{3})^3$ is one of the three cubes, each less than unity to be subtracted from 3, and the other two are to be found by dividing the excess of $(1 + n)^3$ above $p^3 n^3$ into two cubes, after which the numbers sought in the question are to be found by the help of these two last cubes and $(1 - \frac{n}{3})^3$.

It is worthy of remark that in this solution of Fermat, the assumption of $n - 1$ for the side of the square, and $1 - \frac{1}{3}n$ and $1 + n$ for the sides of the cubes to be subtracted from the number three, is of the same nature as Diophantus's assumption of $2\frac{1}{4}$ for the side of the square, i. e. it is justified by its success or founded upon trials or ratiocinations not expressed, it is the known case of $3 - x^2 = y^3 + z^3$, i. e. $3 - 1 = 1 + 1$.

We have given the solutions of Diophantus and Fermat both of them sufficiently brief, but by no means obscure or imperfect: we shall add that of Van Ceulen, remarkable for its fulness and the smallness of the numbers compared with the enormous ones of Bachet, and as Van Ceulen's name occurs seldom but as one of those who had carried the approximation to the quadrature of the circle the

farthest, our readers may like to see a specimen of his abilities in another kind of calculation; and Schooten's Exercitationes Mathematicæ being a work of rare occurrence, we shall translate the 13th section of the 5th book, which is wholly occupied with this problem.

A translation of the 13th section of the 5th book of the Exercitationes Mathematicæ, of Franciscus à Schooten, 4to. printed at Leyden, 1657.

Question the 19th of the 5th book of Diophantus's Arithmetic.

To find three numbers, so that each of them being severally subtracted from the cube of their sum, the three remainders may be cubes.

Of this question the solution given by Diophantus is somewhat difficult. A different one was invented by that skilful arithmetician Ludolphus à Collen, my predecessor in the mathematical chair, in this university. This, judging it will be neither unpleasant nor out of place, I shall here briefly explain, such as I was able to gather it from his own letters, by the favour of the celebrated D. Nicolaus Huberti a Persiin, who was very intimate with him.

For the sum of the numbers sought I put $4N$, and the cube of the sum will be $64N^3$. Then by putting for the first number $56N^3$, for the second $37N^3$, and for the third $63N^3$: there will be, each of them being subtracted from $64N^3$, for the first remainder $8N^3$, for the second remainder $27N^3$, and for the third remainder N^3. Which are all cubes.

Moreover, $156N^3$, the sum of $56N^3$, $37N^3$, and $63N^3$, will be equal to $4N$. Divide each of these equals by $4N$ and we get $39N^2 = 1$.

Now, if this number 39 were a square, the problem would be solved. But since it is not a square number, for the sides of the coefficients of the cube remainders put, $n-1$ for the side of the first, $4-n$ for the side of the second, and 2 for the side of the third. The cubes of which, $n^3 - 3n^2 + 3n - 1$, $64 - 48n + 12n^2 - n^3$, and 8 being each severally subtracted from 64, leave the numbers $65 + 3n^2 - n^3 - 3n$, $48n + n^3 - 12n^2$ and 56. But since the sum of these three numbers is $121 + 45n - 9n^2$: $121 + 45n - 9n^2$ must be a square. I put for its side $11 + n$, then $121 + 22n + n^2 = 121 + 45n - 9n^2$, and n comes out equal to $\frac{23}{10}$. But I supposed the side of the coefficient of the first cube remainder equal to $n-1$, the second to $4-n$, and the third to 2. These sides therefore are $\frac{13}{10}$, $\frac{17}{10}$, and 2. The cubes of these are $\frac{2197}{1000}$, $\frac{4913}{1000}$, and 8, which subtracted from 64 leave the numbers $\frac{61803}{1000}$, $\frac{59087}{1000}$, and 56. Three numbers are therefore found, which severally

subtracted from 64 leave cubic remainders, and being added together make a square.

Hence, $4N$ being put as at first for the sum of the numbers sought, and $64N^3$ for its cube, I now put for the first number $\frac{61903}{1000}N^3$, $\frac{59097}{1000}N^3$ for the second and $56N^3$ for the third, their sum $\frac{176880}{1000}N^3 = 4N$. That is $\frac{17689}{400}N^3 = 1$. Whence $N = \frac{20}{133}$ and $N^3 = \frac{8000}{2352637}$. Which multiplied by $\frac{61903}{1000}$ gives $\frac{494424}{2352637}$ for the first number, and multiplied by $\frac{59097}{1000}$ gives $\frac{472696}{2352637}$ for the second number, and lastly multiplied by 56, gives $\frac{448000}{2352637}$ for the third number, and it is plain that the sum of these $\frac{1415120}{2352637}$ or $\frac{80}{133}$ is equal to $4N$. From the cube of which $\frac{512000}{2352637}$ if you subtract $\frac{494424}{2352637}$, $\frac{472696}{2352637}$, and $\frac{448000}{2352637}$, there will remain the cubes $\frac{17576}{2352637}$, $\frac{39304}{2352637}$ and $\frac{64000}{2352637}$. The sides of which are $\frac{26}{133}$, $\frac{34}{133}$, $\frac{40}{133}$. It is plain therefore that the numbers are rightly found. Cujus rei soli Deo debetur gloria.

It is to be remarked, that this question is the same with the 68th of those hundred questions, which the forementioned arithmetician Ludolphus à Collen, proposed to be solved in his Dutch work on the circle; the 69th of which is of the same kind, viz. to find four numbers so that if from the cube of their sum, they be severally subtracted, the remainders may be cubes.

Since the solution of this depends upon the same artifices as the foregoing. I think it sufficient to put down here only the numbers as they were found by Nicolaus Huberti.

They are as follows: $\frac{867160}{4657463}$, $\frac{787400}{4657463}$, $\frac{13527640}{125751501}$ and $\frac{14097528}{125751501}$; or

$\frac{12172736}{64481201}$, $\frac{11296152}{64481201}$, $\frac{9112168}{64481201}$, $\frac{4724776}{64481201}$.

He found too, besides those of Ludolphus, other three numbers, viz. $\frac{15817515000}{86526634967}$, $\frac{9568152000}{86526634967}$, and $\frac{8925120000}{86526634967}$.

Who also found, to answer the question as proposed for two numbers only, the following numbers, $\frac{87}{125}$ and $\frac{63}{125}$, or $\frac{175608}{493039}$ and $\frac{173888}{493039}$, or these $\frac{105045092893}{69214679736}$ and $\frac{14854859747}{69214679736}$, and so of others.

A solution to this question by Dr. Pell, very nearly the same as

this, and in no respect better, except putting 11 + mn for the side of the square to which $121 + 45n - 9n^2$ is to be equated, may be found in Dr. Wallis's Algebra, Eng. edit. 1685, p. 219, and in Kirkby's Arithmetic (4to. 1735) page 57, prob. 86, part 6. But he, there at least, finds no other numbers than those of Van Ceulen : He has, however, a very quaint remark, if you suppose $m = 2$ or $m = 0$, one or others of the cubic remainders will come out negative. "So that you will think it was only Ludolphus's good hap to steer between those two rocks of negation, and so to light upon such a proposition, as afforded him an answer, wherein none of his numbers fall under 0. Which might give him occasion to conclude as he did, Constat ergo numeros rite esse inventos. Cujus rei soli deo debetur gloria."

A little attention to Van Ceulen's suppositions will shew that after they are made the problem is in fact solved, for if for the sum of the numbers we put $a^2 N$, and for the numbers themselves $(a^6 - b^3)N^3$, $(a^6 - c^3)N^3$, $(a^6 - d^3)N^3$ it will appear as before, that the sum of these coefficients must be a square number, $i.\ e.\ 3a^6 - b^3 - c^3 - d^3 = \square$. To resolve this he puts $b = n - 1$, $c = a^2 - n$, $d = a$, and it results that $2a^6 - a^3 + 1 - 3(a^2 - 1)n^2 + 3(a^4 - 1)n$, must be a square; that this may be effected it is necessary that $2a^6 - a^3 + 1$ be a square, and upon *trial* 2 the least number except one is found to succeed $2.2^6 - 2^3 + 1$ being equal to 121 the square of 11. So that it was perhaps the number 4 that was the cause of Ludolphus's rejoicing. It has been found by trial that the equation $3a^6 - b^3 - c^3 - d^3 = \square$ may be satisfied by $a = 3$, $b = 2$, $c = 3$, and $d = 6$, for then the expression on the left becomes $1936 = (44)^2$ and from these numbers are gotten the least numbers that have yet been found to answer the original question,

viz. $\frac{19467}{85184}$, $\frac{19954}{85184}$, $\frac{13851}{85184}$.

There is a synthetical answer to this question in Kersey's Algebra, vol. 2. page 111, his numbers are $\frac{3045679}{10793861}$, $\frac{183753}{10793861}$ and $\frac{5366599}{10793861}$; or

$\frac{1515784}{6331625}$, $\frac{1622600}{6331625}$ and $\frac{1212416}{6331625}$; or $\frac{2937107}{15069223}$, $\frac{2966301}{15069223}$ and $\frac{2981888}{15069223}$; or $\frac{7777016}{43243551}$,

$\frac{8868015}{43243551}$ and $\frac{8857576}{43243551}$; these are less than those of Bachet, even when reduced to their lowest terms by the aforesaid division by 27, and much less than than those of Nicolaus Huberti a Persiin.

Another Solution.

Let x, y, and z be the numbers sought, s their sum, and a, b, c the roots of the three cubes : Then, by the question,

$$s^3 - x = a^3;\ s^3 - y = b^3;\ s^3 - z = c^3.$$
Or, $x = s^3 - a^3;\ y = s^3 - b^3;\ z = s^3 - c^3.$

And by addition $s = 3s^3 - (a^3 + b^3 + c^3) \ldots\ldots\ldots\ldots\ldots\ (1)$

To satisfy this equation, assume

$$s = u^3v, \quad a = (p + q)v, \quad b = (u^2 - q)v, \quad c = (u^2 - p)v,$$

then substituting these values in equation (1), and dividing the result by v, we have

$$u^3 = v^2 \left\{ u^2 (u^4 + 3pu^2 - 3p^2) + 3(u^4 - p^2)q - 3(p + u^2)q^2 \right\}.$$

Therefore in order that u may be a rational number, the coefficient of v^2, that is, $u^2(u^4 + 3pu^2 - 3p^2) + 3(u^4 - p^2)q - 3(p + u^2)q^2$, must be a square. To make this expression a square generally, we must first make $u^4 + 3pu^2 - 3p^2$ a square. Let its root be assumed

$$= u^2 + \frac{m}{n}p, \quad \text{then}$$

$$u^4 + 3pu^2 - 3p^2 = u^4 + \frac{2mp}{n}u^2 + \frac{m^2}{n^2}p^2;$$

and by reduction $p = \dfrac{3n^2 - 2mn}{3n^2 + m^2}u^2$; therefore $u^2 + \dfrac{m}{n}p$, the root

of $u^4 + 3pu^2 - 3p^2$, is $= \dfrac{3n^2 + 3mn - m^2}{3n^2 + m^2}u^2 = $ (suppose to) ku^2;

then the expression to be made a square is $u^6k^2 + 3(u^4 - p^2)q - 3(p + u^2)q^2$.

Let its root be assumed $= u^3k \pm \dfrac{r}{e}q$, then

$$u^6k^2 + 3(u^4 - p^2)q - 3(p + u^2)q^2 = u^6k^2 \pm \frac{2rku^3}{e}q + \frac{r^2}{e^2}q^2;$$

and by reduction $q = \dfrac{3e^2(u^4 - p^2) \mp 2reku^3}{3e^2(p + u^2) + r^2}$;

and therefore $v = \dfrac{eu}{eku^3 + rq}$,

where m, n, r, and e may be taken at pleasure, within certain limits, which are easily determined for any assumed value of u.

The values of v, p, and q being thus found, we have

$$x = \left\{ u^6 - (p + q)^3 \right\} v^3$$

$$y = \left\{ u^6 - (u^2 - q)^3 \right\} v^3$$

$$z = \left\{ u^6 - (u^2 - p)^3 \right\} v^3$$

Ex. 1. Let m, n, r, and e be each $= 1$, and u also $= 1$, then p is $= \frac{1}{4}$, $k = \frac{5}{4}$, $q = \frac{5}{76}$, $v = \frac{76}{100}$, $p + q = \frac{24}{76}$, $u^2 - q = \frac{71}{76}$, $u^2 - p = \frac{57}{76}$;

therefore $x = \dfrac{76^3 - 24^3}{100^3} = \dfrac{425152}{1000000}$.

$$y = \dfrac{76^3 - 71^3}{100^3} = \dfrac{81065}{1000000},$$

$$z = \frac{76^3 - 57^3}{100^3} = \frac{253783}{1000000}.$$

Ex. 2. Let $e = 2$, and the rest as before, then p is $= \frac{1}{4}$, $k = \frac{2}{5}$, $q = \frac{24}{64}$, $v = \frac{165}{185}$, $p + q = \frac{64}{64}$, $u^2 - q = \frac{39}{64}$, and $u^2 - p = \frac{48}{64}$; therefore $x = \frac{(64^3 - 61^3) \times 2^3}{185^3} = \frac{1545784}{6331625}.$

$$y = \frac{(64^3 - 39^3) \times 2^3}{185^3} = \frac{1622600}{6331625},$$

$$z = \frac{(64^3 - 48^3) \times 2^3}{185^3} = \frac{1212416}{6331625}.$$

Other sets of answers may be obtained by making different assumptions for the values of a, b, and c: thus if we assume

$$a = (u^2 + p - q)v, \quad b = (u^2 - p)v, \quad c = qv,$$

we shall have

$$u^3 = v^2 \left\{ u^2 (u^4 - 6p^2) + 3q(u^2 + p)^2 - 3q^2(u^2 + p) \right\};$$

and to make the coefficient of v^2 a square, we must first make $u^4 - 6p^2$ a square. Let its root be $= u^2 - \frac{m}{n}p$, then

$$u^4 - 6p^2 = u^4 - \frac{2m}{n}u^2 p + \frac{m^2}{n^2}p^2,$$

and by reduction, $p = \frac{2mn}{6n^2 + m^2}u^2$; therefore $u^2 - \frac{m}{n}p$, the root of $u^4 - 6p^2$ is $= \frac{6n^2 - m^2}{6n^2 + m^2}u^2 = $ (suppose to) ku^2; then the expression to be made a square is

$$k^2 u^6 + 3q (u^2 + p)^2 - 3q^2 (u^2 + p).$$

Let its root be assumed $= ku^3 + \frac{r}{e}q$, then

$$k^2 u^6 + 3q(u^2 + p)^2 - 3q^2(u^2 + p) = k^2 u^6 + \frac{2rku^3}{e}q + \frac{r^2}{e^2}q^2;$$

and by reduction $q = \frac{3e^2(u^2 + p)^2 - 2rkeu^3}{3e^2(u^2 + p) + r^2};$

therefore $v = \frac{eu}{eku^3 + rq},$

and $x = \left\{ u^6 - (u^2 + p - q)^3 \right\} v^3,$

$$y = \left\{ u^6 - (u^2 - p)^3 \right\} v^3,$$

$$z = \left\{ u^6 - q^3 \right\} v^3.$$

Ex. Let m, n, r, and e be each $= 1$, and u also $= 1$; then

$p = \frac{2}{7}$, $k = \frac{4}{7}$, $q = \frac{111}{343}$, $u^2 + p - q = \frac{111}{343}$, $u^2 - p = \frac{110}{343}$, and $v = \frac{112}{343}$; hence

$$x = \frac{238^3 - 133^3}{343^3} = \frac{11128635}{40353607},$$

$$y = \frac{238^3 - 170^3}{343^3} = \frac{8568\ 72}{40353607},$$

$$z = \frac{238^3 - 173^3}{343^3} = \frac{8303555}{40353607}.$$

Dr. Pell, as appears from Chapter LIX. of Dr. Wallis's Algebra, satisfies the equation $s = 3s^3 - (a^3 + b^3 + c^3)$ by assuming

$s = 4n$, $a = p - n$, $b = 4n - p$, and $c = 2n$,
then he gets $4n = 121n^3 + 45pn - 9p^2n$,
or dividing by n, $4 = 121n^2 + 45pn - 9p^2$.

He then assumes $11n + p = 2$, and by squaring and reducing finds

$p = \frac{23}{10}n$. But since p is $= 2 - 11n$; $\frac{23n}{10} = 2 - 11n$;

whence $n = \frac{20}{133}$, and $p = \frac{46}{133}$; therefore

$$a = p - n = \frac{46 - 20}{133} = \frac{26}{133},$$

$$b = 4n - p = \frac{80 - 46}{133} = \frac{34}{133},$$

$$c = 2n \qquad\qquad = \frac{40}{133}.$$

Whence $x = (4n)^3 - a^3 = \dfrac{80^3 - 26^3}{133^3} = \dfrac{494424}{2352637},$

$$y = (4n)^3 - b^3 = \frac{80^3 - 34^3}{133^3} = \frac{472696}{2352637},$$

$$z = (4n)^3 - c^3 = \frac{80^3 - 40^3}{133^3} = \frac{448000}{2352637}.$$

Dr. Pell then shews how other answers may be derived from the equation $4 = 121n^2 + 45pn - 9p^2$, by assuming $11n + qp = + 2$ or -2, and is at great pains to determine between what limits q must be taken, so as always to bring out positive answers, but his solution is much too long to be inserted here.

Different sets of answers may be obtained by assuming other square numbers, instead of 4, for the coefficient of n in the value of s. Thus if we assume $s = 9n$, $a = p - n$, $b = 9n - p$ and $c = 3n$, we have

$$9 = 1432n^2 + 240pn - 24p^2.$$

But here the coefficient of n^2 is not a square as in Dr. Pell's solution; therefore we must find by trial a case in which $1432n^2 + 240pn - 24p^2$ shall be a square; and this evidently happens when $p = 3n$; for the expression then becomes $(1432 + 720 - 216)n^2 =$

$1936n^2 = 44^2n^2$. Hence $9 = 44^2n^2$, or $n = \frac{3}{144}$; therefore $s = \frac{27}{44}$, $a = \frac{6}{44}$, $b = \frac{18}{44}$, $c = \frac{9}{44}$;

$$x = \frac{27^3 - 6^3}{44^3} = \frac{19467}{85184},$$

$$y = \frac{27^3 - 18^3}{44^3} = \frac{13851}{85184},$$

$$z = \frac{27^3 - 9^3}{44^3} = \frac{18954}{85184}.$$

Having obtained one solution by making $p = 3n$, we may find as many other solutions as we please by assuming $p = n(3 + r)$; for then

$$9 = n^2(44^2 + 96r - 24r^2),$$

and we have only to make $44^2 + 96r - 24r^2$ a square by assuming its roots $= 44 + sr$.

VI.   QUESTION 52, *by Mr*. Tho. Pointin.

A malster has two cubical bins which hold together 119·998 bushels, and the side of the greater exceeds that of the less by 43 inches : What is the side of each ?

Answered by Mr. Ed. Moore. .

The side of the biggest cistern . . . . 62·9999314825 inches.
The side of the lesser. 19·9999314825 inches.
The biggest. . . . . 116·2793 bushels.
The lesser . . . . 3·7187 bushels.

Which may be found thus : Let $43 =$ difference of the sides, be $= d$, and the side of the greater $= a$, the side of the lesser shall be $= a - d$, and the sum of their cubes $= 2aaa - 3da^2 + 3d^2a - d^3 = 119\cdot9998$ bush. $\times$ 2150·42, the solid inches in a Winchester bushel $= 258046\cdot099$ inches, and the value of a will be found $= 63$ proxime ; consequently $a - d = 20$ inches. Then $\dfrac{63^3}{2150\cdot42} = 116\cdot278$ bush. and $\dfrac{20^3}{2150\cdot42} = 3\cdot720$ bushels, but their sum $= 119\cdot998$ required.

Additional Solution.

This question may be otherwise more simply solved thus : Let z denote the half sum of the sides of the cubes, and $d = 21\frac{1}{2}$ the half difference ; then will the sides themselves be $z + d$ and $z - d$; and the sum of their cubes is $2z^3 + 6d^2z = 119\cdot998 \times 2150\cdot42$ (the inches in a bushel), or $z^3 + 1386\frac{1}{4}z = 59\cdot999 \times 2150\cdot42$. Hence $z = 41\frac{1}{2}$ nearly ; and consequently the sides $= 63$ and 20.   H.

PRIZE QUESTION.

What were the position of the four satellites of Jupiter on Octo-
ber 7th, 1716, at ten in the evening, with respect to the east or west
sides of the planet?

This question was answered by Mr. T. Wright, A. B. *and*
Mr. Ronkesley.

Solution.

It is evident that the difference between the geocentric place of Jupi-
ter and the place of each of his satellites, for the given time, will give
their distances from him in signs, degrees, &c. which may be turned
into semi-diameters of Jupiter.   To do all which is fully taught in
most books of astronomical tables.   These calculations made for the
proposed time will give the distance in semi-diameters and tenths of
Jupiter thus ; *viz.* 1st satellite = 0·8 W. the second sat. = 7·9 W.
the 3d sat. = 15·0 W. and the 4th sat. = 2·2 E.   Which being drawn
from a scale of equal parts will appear thus
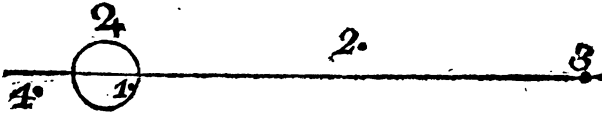
Where the configurations are adapted to the explanation given in the
Nautical Almanac.   H.

Questions proposed in 1717, *and answered in* 1718.

I.   QUESTION 53, *by Mr.* Tho. Dodd.

A merchant sold sixteen diamonds at prices increasing in geome-
trical progression. He got a penny for the first, and 14348907 pence
for the last.   What did the sixteen diamonds come to in all, and
what is the common multiplier or ratio of the progression?

Answered by Mr. J. Peirce.

$\sqrt[15]{14348907} = 3$ the ratio.   Let p be = first term = 1, $s =$

second term $= 3$, $u =$ last $= 14348907$, and $z =$ sum of all the terms. Now $z - u =$ sum of all the antecedents; $z - p =$ sum of all the consequents. Therefore $p : s :: z - u : z - p$; and by equality $pz - pp = sz - su$: by transposition, and division, $z = \dfrac{3u - 1}{2} = 21523360$ pence $= 89680l.$ 13s. 4d.

II.   QUESTION 54, *by Mr.* Robert Beales, *of Lynn.*

Two footmen set out (to meet each other), at the same time, from the cities of London and Norwich, distant 99 miles. The footman from London goes three miles the first day, five miles the second, seven miles the third, &c. increasing two miles each day. The footman from Norwich travels the first day a number of miles equal to one-sixth part of the number of days in which they meet, increasing each succeeding day's journey by three miles. In how many days will they meet, and how many miles will each have travelled?

Answered by Mr. R. Hull.

They would meet in 6 days: The footman from Norwich travelled 48, and the Londoner 51 miles.

Solution.

Putting x for the number of days they travelled, or the number of terms in each series, of which the two common differences are 2 and 3; the corresponding first terms will be 3 and $\frac{1}{6}x$,

the last terms $= \begin{cases} 2 \cdot (x-1) + 3 = 2x + 1 \\ 3 \cdot (x-1) + \frac{1}{6}x = 3\frac{1}{6}x - 3, \end{cases}$

the sum of each pair of extremes $= \begin{cases} 2x + 4 \\ 3\frac{1}{6}x - 3, \end{cases}$

and the sum of each series, or the miles each travelled $= \begin{cases} x^2 + 2x \\ \frac{5}{6}x^2 - \frac{3}{2}x. \end{cases}$

Then by taking the sum of these two sums, we have $\frac{8}{6}x^2 + \frac{1}{2}x = 99$; and hence $x = 6 =$ the number of days, and $\begin{cases} x^2 + 2x = 48 \\ \frac{5}{6}x^2 - \frac{3}{2}x = 51 \end{cases} =$ the miles each travelled.   H.

III.   QUESTION 55, *by* Xantippe, *of Helmsley, Yorkshire.*

A legacy of 229l. 13s. 4d. is to be divided amongst three persons, M, A, N, in the following manner, namely, as often as M takes $5\frac{3}{4}l.$ A is to take $4\frac{3}{7}l.$; and as often as A takes $4\frac{1}{5}l.$ N is to take $3\frac{2}{4}l.$ What are the respective shares?

Answered by Mr. J. Simmons.

911. 11s. $41\frac{1673}{2842}$d. is the share for M.
72   19 $10\frac{1693}{2842}$ is what A ought to take.
65   2 $0\frac{2058}{2842}$ is the part for N.

Solution.

The numbers $5\frac{5}{9}$ and $4\frac{3}{7}$, or $\frac{50}{9}$ and $\frac{31}{7}$ are in proportion as 350

and 279; and the numbers $4\frac{1}{9}$ and $3\frac{2}{3}$, or $\frac{37}{9}$ and $\frac{11}{3}$ are to each other

as 37 to 33. Now, $279 : 350 :: 37 : \frac{37 \times 350}{279}$; and hence the

three shares must be in proportion, as the three numbers $\frac{37 \times 350}{279}$,

37, and 33: or as 37×350, 37×279, and 33×279; that is, as the
numbers 12950, 10323, and 9207: whose sum is 32480.

Whence, by proportion, we shall have as 32480 : 2291. 13s. 4d.

$$::\begin{cases} 12950 : 91l. \ 11s. \ 4\frac{294}{406}d. \ = \text{M's share,} \\ 10323 : 72 \quad 19 \quad 10\frac{239}{406} \ = \text{A's share,} \\ 9207 : 65 \quad 2 \quad 0\frac{279}{406} \ = \text{N's share.} \quad \text{H.} \end{cases}$$

IV.   QUESTION 56, *by Mr.* W. Beriffe, *of St. Ives.*

A surveyor measured a field in the form of a trapezium, and found
its area 2 acres 144 poles, and its sides 4, 6, 7, and 5 chains. What
is the length of the greatest diagonal?

Answered by Mr. Ja. Peirce.

Let BC $=$ 4 $= b$, AD $= 7 = g$,
   CD $=$ 6 $= c$, AB $= 5 = f$,
CD $+$ BC $= 10 = e$, AD $+$ AB $= 12 = h$,
CD $-$ BC $= 2 = d$, AD $-$ AB $= 2 = i$.

The area 2 a. 3 r. 24 per. $= 29$ sq. cha. $= a$.
Quere diagonal BD $= y$.

(1). $y : e :: d : de \div y = $ DE $-$ EB;
and $y : h :: i : hi \div y = $ DF $-$ FB.

(2). $(yy + de) \div 2y = $ DE, and $(yy + hi) \div 2y = $ DF.

(3). $\sqrt{\dfrac{4c^2y^2 - y^4 - 2dey^2 - d^2e^2}{4y^2}} = $ CE:

$\sqrt{\dfrac{4g^2y^2 - y^4 - 2hiy^2 - h^2i^2}{4y^2}} = $ AF.
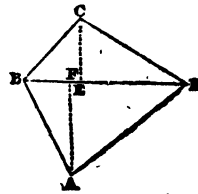
(4). These two last × by $\frac{1}{2}y$, and added together give the area.

(5). Then by extracting the square root of the denom. and multiplying a by 4, you will have

$$\sqrt{(4c^2y^2 - y^4 - 2dcy^2 - d^2e^2)} +$$
$$\sqrt{(4g^2y^2 - y^4 - 2hiy^4 - h^2i^2)} = 4a.$$

(6). By squaring both sides of the last equation, and transferring the rational terms to one side, you will have (after making $4c^2 - 2ed = k$, $- ddee = - l$; $4g^2 - 2hi = m$, $- hhii = - n$;) $2\sqrt{((ky^2 - y^4 - l) \times (my^2 - y^4 - n))} = 16aa - ky^2 - my^2 + 2y^4 + l + n$.

Let $16a^2 + l + n = p$, and $- ky^2 - my^2 = - qy^2$.

(7). By squaring both sides of this last (p and $- q$ rightly placed) you will have $4kmy^4 + 4ly^4 + 4ny^4 - 4lmy^2 - 4kny^2 - 4my^6 - 4ky^6 + 4y^8 + 4ln = p^2 + 4py^2 + q^2y^4 - 2pqy^2 + 4y^8 - 4qy^6$.

(8). But $4y^8 = 4y^8$ is destroyed; also by supposition $m + k = q$. Therefore $- 4my^6 - 4ky^6 = - 4qy^6$, which will be destroyed. Lastly, by transposing the known quantities to one side, and the unknown to the other, you will have $4kmy^4 + 4ly^4 + 4ny^4 - 4py^2 - q^2y^4 + 2pqy^2 - 4lmy^2 - 4kny^2 = p^2 - 4ln$.

Let $4km + 4l + 4n - 4p - qq = r$; $2pq - 4lm - 4kn = s$, and $p^2 - 4ln = t$. Then $ry^4 + sy^2 = t$.

From whence will arise this theorem.

$$y = \sqrt{\left(\sqrt{\left(\frac{ss}{4rr} + \frac{t}{r}\right)} - \frac{s}{2r}\right)} = 7 \cdot 682 \text{ chains, the longer}$$

diagonal; the other is $7 \cdot 554$ chains.

Additional Solutions.—1st. Algebraically.

Put $AB = a$, $BC = b$, $CD = c$, $AD = d$, angle $BAD = \varphi$, angle $BCD = \theta$, and the double area $= e$. Then by trigonometry

$$a^2 + d^2 - 2ad \cos \varphi = BD^2 = b^2 + c^2 - 2bc \cos \theta,$$

therefore $2ad \cos \varphi - 2bc \cos \theta = a^2 + d^2 - b^2 - c^2$.

Again, double the area of the triangle $BAD = ad \sin \varphi$, and double the area of the triangle $\qquad BCD = bc \sin \theta$; therefore, putting $2r = a^2 + d^2 - b^2 - c^2$, we have

$$ad \sin \varphi + bc \sin \theta = e$$
$$\text{and} \quad ad \cos \varphi - bc \cos \theta = r.$$

Hence $a^2d^2 \sin^2\varphi = (e - bc \sin \theta)^2$,

and $a^2d^2 \cos^2\varphi = (r + bc \cos \theta)^2$,

therefore by addition, and remarking that $\sin^2\varphi + \cos^2\varphi = 1$, and $\sin^2\theta + \cos^2\theta = 1$, $a^2d^2 = (e - bc \sin \theta)^2 + (r + bc \cos \theta)^2 = e^2 - 2ebc \sin \theta + r^2 + 2rbc \cos \theta + b^2c^2$,

or, $r \cos \theta - e \sin \theta = \dfrac{a^2d^2 - b^2c^2 - e^2 - r^2}{2bc} = m$.

Whence $r \cos \theta - m = e \sin \theta$;

or, $r^2 \cos^2\theta - 2rm \cos \theta + m^2 = e^2 \sin^2\theta = e^2 - e^2 \cos^2\theta$,

therefore $(r^2 + e^2) \cos^2 \theta - 2r m \cos \theta = e^2 - m^2$,

and, dividing by $r^2 + e^2$ and completing the square,

$$\cos \theta = \frac{1}{r^2 + e^2} \times \left\{ e \sqrt{(r^2 + e^2 - m^2)} + rm \right\},$$

and the diagonal BD $= \sqrt{(b^2 + c^2 - 2bc \cos \theta)}$.

Or since $(bc \sin \theta)^2 = (e - ad \sin \varphi)^2$

and $(bc \cos \theta)^2 = (r - ad \cos \varphi)^2$,

we have as before

$$e \sin \varphi + r \cos \varphi = \frac{a^2d^2 - b^2c^2 + e^2 + r^2}{2ad} = n,$$

therefore $\cos \varphi = \dfrac{1}{r^2 + e^2} \times \left\{ r \sqrt{(r^2 + e^2 - n^2)} + rn \right\}$,

and the diagonal BD is then $= \sqrt{(a^2 + d^2 - 2ad \cos \varphi)}$.

Here it may be remarked, that when the value of e is such as to make either m^2 or n^2 greater than $r^2 + e^2$, the part under the radical becomes negative, and consequently the problem does not then admit of a solution. Therefore the limit of possibility, or the case in which the area is the greatest possible, will be when m^2 and n^2 are each equal to $r^2 + e^2$. The values of m and n are then equal, but have a different sign, and the above expressions for cos θ and cos φ give

$$\cos \theta = \frac{rm}{r^2 + e^2} \text{ or } \frac{r}{\sqrt{(r^2 + e^2)}}; \qquad \cos \varphi = \frac{rn}{r^2 + e^2} \text{ or } \frac{-r}{\sqrt{(r^2 + e^2)}};$$

therefore cos θ is $= - \cos \varphi$, and consequently the one angle is the supplement of the other, and the trapezium is inscribed in a circle.

Again because $m = - n$, $\dfrac{a^2d^2 - b^2c^2 - e^2 - r^2}{2bc} = - \dfrac{a^2d^2 - b^2c^2 + e^2 + r^2}{2ad}$;

therefore by reduction $e^2 = (ad + bc)^2 - r^2$, or $e = \sqrt{\left\{ (ad + bc)^2 - r^2 \right\}}$

Hence, when the four sides are given, if the double area be greater than $\sqrt{\left\{ (ad + bc)^2 - r^2 \right\}}$ the problem is impossible; and it is curious enough that this should be the case in the Diary question, which it certainly is, for e is then $= \sqrt{\left\{ (5 \times 7 + 4 \times 6)^2 - 11^2 \right\}}$ $= \sqrt{(59^2 - 11^2)} = \sqrt{3360}$ which is less than 58. The original solution cannot therefore be correct.

The common rule for finding the area of a trapezium capable of being inscribed in a circle, when the four sides are known, may be deduced immediately from the above expression for e, by considering that $(ad + bc)^2 - r^2$ is $= (ad + bc + r)(ad + bc - r) = \frac{1}{4}(2ad + 2bc + a^2 + d^2 - b^2 - c^2)(2ad + 2bc - a^2 - d^2 + b^2 + c^2) = \frac{1}{4}(a + d + c - b)(a + d - c + b)(b + c + a - d)(b + c + d - a)$.

2nd. *Geometrically.*

Suppose ABCD is the trapezium required, and
from D draw DF perpendicular to BC and DE
perpendicular to BA. Then Euclid 13, II.
$BC^2 + CD^2 + 2BC \cdot CF = BD^2 = BA^2 + AD^2 - 2BA \cdot AE$;
hence $2BC \cdot CF + 2BA \cdot AE = BA^2 + AD^2 - BC^2 - CD^2$.

Also BC · DF = twice the △ BCD, and BA · DE
= twice the △ BDA; therefore BC · DF + BA · DE
= twice the given area. Take HE a fourth pro-
portional to AB, BC and CF, and make the triangle EKH similar to DCF;
then $2BC \cdot CF$ will be = $2BA \cdot EH$, and BC · DF = AB · EK; therefore
$2BA$ (EH + AE) = $2BA$ · (AH) = $BA^2 + AD^2 - BC^2 - CD^2$, a given space,
and BA (EK + ED) = BA · DK = twice the given area;
therefore AH and DK are given lines.

Make HI perpendicular to AH and = to DK and join ID, then
KHID will be a parallelogram, I a given point, and ID = HK a given
line (being a fourth proportional to AB, BC and DC); hence if from
the centres A and I, with distances equal to AD and ID (or HK), two
circles be described, their intersection will determine the point D,
then if BD be drawn it will be the diagonal required.

The calculation from hence is extremely easy, for in the right-
angled triangle AHI, the side AH is = $(BA^2 + AD^2 - BC^2 - CD^2) \div 2BA$,
the side HI = twice the given area divided by BA; whence AI and
the angle HAI may be found: Again in the triangle AID, the three
sides are given; whence DAI, BAD, become known, and the diagonal
BD may be found.

The construction of the problem is evidently impossible when the
circles described from the centres A and I neither cut nor touch one
another. But when they touch, we have the limits between which the
area must be included to render the construction possible; namely,
the greatest limit when they touch externally, and the least when
they touch internally. In both cases the point D will be in the line
AI, and the triangle ADE similar to the triangle CDF; therefore the
angle BCD will be the supplement of BAD; consequently the trape-
zium will be inscribed in a circle when the area is the greatest; and
will have a re-entering angle when it is the least.

Constructions and other solutions of this question may be seen in
Simpson's Algebra and his Select Exercises.   J. L.

V.   QUESTION 57, *by Mr.* J. Symmons.

A grass-plot is to be made in the form of an ellipse, whose prin-
cipal diameters are to be marked out in the following manner.
The latitude of the place is 52° 50′ N. and a pole 6 feet 8 inches
long having a small ball upon the top, is to stand perpendicular to
the plane of the horizon at the centre of the grass-plot; one of the
semidiameters is to be the shadow of the pole at noon on the second
of March, and the other semidiameter is to be the shadow of the pole
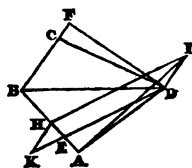
on the eleventh of June, when the sun is due west. There is then to
be an elliptical step of stone five inches thick, concentrical with the
grass-plot, and having the extremities of its principal axes equally
distant from the extremities of the principal axes of the grass-plot.
Upon the step are to be placed four brazen lions couchant, support-
ing a cube, which is again to support a dodecaedron, and this again,
a sphere. The contents of the cube, dodecaedron, and sphere to
have the ratio of 6, 3, and 1 respectively, and to contain as many
cubic inches as one third of the area of the grass-plot contains square
inches. Required the sides of the cube and dodecaedron, as also the
diameter of the sphere.

Answered by Ja. Peirce, Wm. Whitworth, Jos. Wilkinson, *and*
Moses Bleathman.

Mar. 2. Sun's longitude 22° 56′ 34″ ♓. Declin. 2° 48′ 28″ S.
The sun's meridian altitude, 34° 21′ 32″.
Length of the shade, 117·056 inches; which doubled, gives the
the conjugate diameter of the oval 234·112 inches.
June 11. Sun's longitude.......................♋ 0° 55′ 41″
    Declination north......................... 23 29 48
    Sun's altitude east or west 30 1 16
    Length of the shadow .... 138·4 in.
    Double is the transverse diameter....... 276·8
    The area of the oval ....50916·0155
    $\frac{1}{3}$ Taken for the 3 solid bodies ....16972·0051
    $\frac{2}{3}$ For the step ....33944·0102
    Transverse diam. of step .... 116·716
    The conjugate diam. of the step ...... 74·028
    The side of the cube ... . 21·675
    Side of the dodecaedron.............. 8·726
    Diameter of the globe................ 14·799

Solution.

The method of solving this question is thus :
By astronomical tables find the sun's place or longitude to the
two given times; and the obliquity of the ecliptic or the greatest
declination for the same : then by spherics, it will be, as radius
: sine long. : : sine obliq. : sine of the present declinations. In the
first case, the declin. taken from the colat. leaves the meridian
altitude; and in the other, we shall have, by spherics, as s. lat. : s.
obliq. : : radius : s. altitude. These altitudes being thus computed,
we shall suppose to be 34° 21′ 32″, and 30° 1′ 16″, as in the
original solution above.
Then, by plain triangles, as 1 (rad.) : 80 (length of staff) : :
cotang. alt. : length of the shade; consequently 160 × 1·4622874
(cotang. 34° 21′ 32″) = 234·034176 inches = the conjugate dia-

meter, and $160 \times 1\cdot7305780$ (cotang. 30° $1'$ $16''$) $= 276\cdot89248$ $=$ the transverse diameter ; and hence the area of the green is $276\cdot89248 \times 234\cdot034176 \times \cdot78539$ &c. $= 50895\cdot58$. Then $\frac{1}{3}$ of this being a number expressing the sum of the contents of the three solids, which are in proportion to one another as 1, 3, and 6, whose sum

is 10; we shall have, as $10 : \dfrac{508905\cdot58}{3}$

$$:: \begin{cases} 1 : & 1696\cdot519 = \text{the content of the sphere} \\ 3 : & 5089\cdot558 = \text{the content of the dodecaedron} \\ 6 : & 10179\cdot116 = \text{the content of the cube} \end{cases} : \text{and, by}$$

mensuration $\sqrt[3]{10179\cdot116} = 21\cdot6722 =$ the side of the cube,

$$\sqrt[3]{\dfrac{5089\cdot558}{7\cdot66311896}} = 8\cdot7248 = \text{the side of the dodecaedron,}$$

$$\sqrt[3]{\dfrac{1696\cdot518}{\frac{1}{6} \times 3\cdot14159\&c.}} = 14\cdot7975 = \text{the diameter of the sphere.}$$

Again, for the diameters of the inner ellipse, or stone step, whose solidity is $\frac{2}{3}$ of the area of the outer one, and the difference of its diameters $=$ ($276\cdot89248$ — $234\cdot03418$ $=$) $42\cdot8583$; let $x =$ half the sum of its diameters, then the diameters will be $x \pm$ $21\cdot42915$ $\left(=\dfrac{42\cdot8583}{2}\right)$; and their product $x^2 - 21\cdot41915^2$ (per quest.) $= 276\cdot89248 \times 234\cdot03418 \times \frac{2}{3} \times \frac{1}{2} = 8640\cdot304$; hence $x = \sqrt{(21\cdot42915^2 + 8640\cdot304)} = 95\cdot3914$, and $95\cdot3914 \pm 21\cdot4291 = 116\cdot8205$ and $73\cdot9623 =$ the diameters of the inner ellipse. H.

VI.   QUESTION 58, *by the Lady* Mer. Heyshot.

A lady observes, that if the age of the younger of her two sons is subtracted from the square of the age of the elder, the remainder will be 425 : but if the age of the elder is subtracted from the square of the age of the younger the remainder will be 235.   What is the age of each ?

Answered by Mr. Moses Bleathman.

The ages are 21 and 16 years.

Solution.

Putting x and y for the two ages, we have, per question, $x^2 -$ $y = 425$, and $y^2 - x = 235$; hence $y^2 = 235 + x = (x^2 - 425)^2$, or $x^4 - 850x^2 - x = -180390$; in which equation $x = 21$; and then $y = x^2 - 425 = 16$. H.

PRIZE QUESTION.

There is a tetraedron standing on level ground whose content is

five cubic feet; and there is also a hollow prism standing near it, capable of holding thirty-six ale gallons. The prism is open at the ends, which are equilateral triangles, and the height of the prism is equal to the side of the triangle which constitutes its end or base. Now if the hollow prism is put upon the tetraedron so as exactly to fit the end, what will be the height of the end of the prism above the base of the tetraedron? And suppose another pyramid put into the other end of the prism so as exactly to fit the same, and its height such that the vertex thereof shall just meet that of the tetraedron. What quantity of liquor will fill the space of the prism which is left vacant?

Answered by Mr. Char. Mason.

<div align="right">Inches.</div>

The side of each triangle of the tetraedron ....41·853
The perpendicular altitude..................34·17283136
The side and altitude of the prism...........28·6209
Consequently the side of the base of that part } 28·6209
  of the tetraedron which is within the prism }
Its perpendicular altitude....................23·3688
Its content in ale-gallons....................9·79793
The perpendicular altitude of the pyramid ....5·25204
Its content in ale-gallons....................2·20203
The content of the two pyramids.............11·99999
Content of the prism when placed according } 24·0000
  to the import of the question in ale-gallons }
The elevation of the prism ....10·8039

Solution.

Since the two ends of the prism are filled up by two pyramids whose vertexes meet within it, the height of the two together is just equal to the altitude of the prism; and since a pyramid is $= \frac{1}{3}$ of a prism of the same altitude and base, it is evident that $\frac{2}{3}$ of the prism will be filled with liquor, that is, it will contain 24 gallons : which is one part of the demand. The other is to find the additional elevation of the prism, which, being evidently $=$ the difference between the altitude of the tetraedron and of the part of it which is let into the prism, will be found thus.

By the Schol. page 403 Mensuration, $12 \sqrt[3]{(6 \times 5 \sqrt{2})} =$ $12 \sqrt[3]{(30 \sqrt{2})} = 41·853$ inches $=$ the side of the whole tetraedron ; and by pages 81 and 145, we shall have $\sqrt[3]{\dfrac{36 \times 282 \times 4}{\sqrt{3}}} =$

$6 \sqrt[3]{\dfrac{188}{\sqrt{3}}} = 28·62092 =$ the side of the prism $=$ the side of the

tetraedron which is within the prism. And it is the difference of the altitudes of these two which is required. Now, from the demon-

<div align="center">E 4</div>

stration, page 403 Mensuration, it appears that the side of a tetrae-
dron is to its altitude, as 1 to $\sqrt{\frac{2}{3}}$; and since the difference between
41·85300 and 28·62092 the sides of the two tetraedrons is 13·23208,
we shall have 13·23208 $\sqrt{\frac{2}{3}}$ = 10·80394 = the additional elevation
required. H.

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Questions proposed in 1718, and answered in 1719.

I. QUESTION 59, by Mr. Doidge, of Portsmouth.

Required the side of a regular decagon whose area shall be
163 feet?

Answered by Anna Philomathes.

55·23 inches = to the polygon's side = 4·602 feet.

Solution.

By mensuration, $\frac{5}{2}\sqrt{(5 + 2\sqrt{5})}$ or 7·6942088 drawn into the
square of the side of a decagon is its area; wherefore the side will be
$\sqrt{\frac{163}{7·6942088}}$ = 4·6027 nearly. H.

II. QUESTION 60, by Mr. Tho. Cary, of Lynn.

There is a circular pond whose area is 5028$\frac{4}{7}$ square feet, in the
middle of which stood a pole 100 feet high; now the pole having
been broken it was observed that the top just struck the brink of
the pond, what is the height of the piece left standing?

Answered by Mr. W. Laughton.

Let $r = 40·008$ the radius of the pond; $h =$ to the piece broke
off; and $p =$ to standing piece. Then $h + p = a = 100$, and
$hh = pp + rr$ by 47 Euclid I. Hence $hh + 2hp + pp = aa$.
And $hh — pp = rr$. By subtracting, $2hp + 2pp = aa — rr$.
Then $p = \dfrac{aa — rr}{2h + 2p} = \dfrac{aa — rr}{2a} = 41·9968$ feet.

Additional Solutions. 1st. Algebraically.

The square of the radius of the circle is $\dfrac{5028\frac{4}{7}}{4 \times ·78539 \text{ &c.}}$ =

$$\frac{8800}{7 \times \cdot78539 \ \&c.} = 1600{\cdot}644, \text{ which put} = a^2; \text{ also put } c = 100$$

the whole height of the pole, and $x =$ the part standing. Then will $c - x$ be the part broken off, and is the hypothenuse of a right-angled triangle whose two legs are a and x; consequently $a^2 + x^2$

$$= (c - x)^2 = c^2 - 2cx + x^2; \text{ and hence } x = \frac{c^2 - a^2}{2c} = 41{\cdot}99678$$

$=$ the length required.

2nd. *Geometrically.*

Here are given the base and the sum of the hypothenuse and perpendicular, of a right-angled triangle; to construct the same. Which will be done thus :

Making AB the radius of the circular pond, and BC the pole perpendicular to it; join A, C; and draw AD making the angle DAC $= \angle$ DCA ; so shall BD be the part standing, and DA or DC the part broke off.—For because of the equality of the $\angle$'s DCA, DAC, their opposite sides DA, DC will be equal. *Q. E. D.* H.
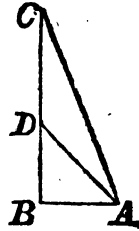
III. QUESTION 61, *by Mr.* Peter Ward.

It is required to fill a magic square, consisting of 100 squares or spaces, that is, 10 in length, and as many in breadth, with numbers in such a manner that the 10 lines in length and the 10 lines in the breadth shall each amount to 505 ; and also the two diagonals shall each amount to 505, and the numbers in no two squares or spaces to be alike.

Answer. There are many different ways of answering or filling this magic square.

The following one is thus filled by Mr. *J. Harris.*

11	92	12	88	14	15	16	84	83	90
100	82	26	27	67	35	59	58	50	1
99	19	75	74	33	66	42	43	51	3
2	20	76	73	34	36	60	57	49	98
4	81	25	28	68	65	41	44	52	97
94	21	77	72	32	37	61	56	48	7
5	80	24	29	69	64	40	45	53	96
6	79	23	30	70	38	62	55	47	95
93	22	78	71	31	63	39	46	54	8
91	9	80	13	87	56	85	17	18	10

These numbers make 505 ten times in the lines, ten times in the columns, and twice diagonally.

Additional Solution.

A general method of filling Magic Squares. By Numericus.

When an arithmetical series of numbers are disposed in the cells of a square, so that the sums in the horizontal, vertical, and diagonal ranks are alike, the square is called a *magic square*. It is not necessary, however, that the series should be a continued one; because it may be disjunct or broken, as 1 2 3 4 9 10 11 12; or 2 4 6 11 13 15 20 22 24, &c.

We shall begin with squares whose roots are odd numbers; the general method of construction being more simple than in cases where the roots are even.

To make a magic square of 25 cells with the series 1, 2, 3, &c.

Put 13 the middle number of the series in the middle cell (*Fig. 1.*); the greatest and least numbers 25 and 1 in the cells next above and below; the root 5 and its complement (21) to 26 (= 25 + 1) in the adjacent cells to the left and right; then fill one diagonal with the numbers in their natural order from 13 both ways; and let the common difference of the numbers in the other diagonal be the root 5.

Then each diagonal will contain $\dfrac{(25 + 1) \times 12\frac{1}{2}}{5} = 65$ the number

required in each rank of the square.

The 3d. or middle columns, vertical and horizontal, are each 26 short of 65; but 8 + 18 (in the diagonal) = 26; consequently 8—1 + 18 + 1, and 8 + 1 + 18—1 respectively make 26; hence if 8—1, 8 + 1, and 18—1, 18 + 1 (or 7, 9, 17, 19) are in the extreme cells, those columns will each contain 65. Now 7, 8, 9 ; 17, 18, 19 have the same constant difference as the numbers in that diagonal to which they are parallel. And 7, 12, 17; and 9, 14, 19 in cells parallel to the other diagonal differ by the root 5, the common difference of the numbers in that diagonal. In this manner we get the numbers for the other cells. Hence to fill a square whose root is an odd number:—Place the middle numbers and fill the diagonals as above directed; then let the other numbers in the cells parallel to the diagonals, have the same respective differences as the numbers in those diagonals. (*Fig. 2.*)

Fig. 1.

11				3
	12	25	8	
	5	13	21	
	18	1	14	
23				15

Fig. 2.

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

Figures 3 and 4 are magic squares whose sides are 3 and 9, constructed in this manner:

Fig. 3.

4	9	2
3	5	7
8	1	6

Fig. 4.

37	78	29	70	21	62	13	54	5
6	38	79	30	71	22	63	14	46
47	7	39	80	31	72	23	55	15
16	48	8	40	81	32	64	24	56
57	17	49	9	41	73	33	65	25
26	58	18	50	1	42	74	34	66
67	27	59	10	51	2	43	75	35
36	68	19	60	11	52	3	44	76
77	28	69	20	61	12	53	4	45

In these squares it is manifest that the opposite columns, equidistant from the centre, may change places, and the columns on the same side of the centre may also be changed, provided a similar variation takes place in the other half of the square.

When the root is *unevenly-even*, as 6, 10, 14, &c. Let the square be that of 6, with the series 1, 2, 3, 4, &c. to 36.

Arrange the numbers in pairs as below.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
36 35 34 33 32 31 30 29 28 27 26 25 24 23 22 21 20 19

Each pair being 37 or $\frac{1}{3}$ of 111 the number required in each rank of the square; or $\dfrac{(36 + 1) \times 18}{6} = 111$.

Let these pairs be regularly placed in a square (*Fig. 5.*) the greater number above the less in the alternate vertical columns. Then each vertical column will be 111; but the horizontal columns are alternately 102 and 120, or the defect in one column equal to the excess in the other. Now the common differences of the two middle numbers, the two next, and the two extremes in the 5th. vertical column from the left, is 9: let these change places and the vertical and horizontal ranks of the square will each be 111 (*Fig. 6.*) The diagonals, however, are 156 and 66, therefore the excess and defect are equal; and since the pairs, and also the columns may be changed at pleasure, the diagonals are easily made equal:

Fig. 5.

1	33	7	27	13	21
36	4	30	10	24	16
2	32	8	26	14	20
35	5	29	11	23	17
3	31	9	25	15	19
34	6	28	12	22	18

Fig. 6.

1	33	7	27	22	21
36	4	30	10	15	16
2	32	8	26	23	20
35	5	29	11	14	17
5	31	9	25	24	19
34	6	28	12	13	18

thus, change $\dfrac{20}{17}$ for $\dfrac{23}{14}$, then let the 2d. vertical column take the

place of the 5th; the 3d that of the 2d; and the 5th the place of the 3d, *(Fig. 7).* The same thing, however may be effected by shifting pairs only. In this manner a square of 10 may be constructed.

Fig. 7.

1	7	22	27	33	21
36	30	15	10	4	16
2	8	20	26	32	23
35	29	17	11	5	14
3	9	24	25	31	19
34	28	13	12	6	18

But in the case of roots unevenly-even, an easier method seems to be that of adding a border to a square. We shall give an example of constructing a magic square of 100 cells, with borders, or *a magic square containing magic squares,* by adding three successive borders to the square of 4.

There are several methods of filling the square of 16 cells; the following, however is easily remembered :—Arrange the 16 numbers in a square thus, *(Fig. 8)*

One diagonal consists of the two extremes of the series with two other numbers equally distant from those extremes; the other of 4 numbers, two and two, which also are respectively at equal distances; consequently the diagonals are each $= (16 + 1) \times 2 = 34 = \dfrac{(16 + 1) \times 8}{4}$ the number required in each rank of the square. Next, 2, 15, and 3, 14 are also equidistants, and their sum $= 34$. The difference of the two middle perpendicular ranks is 4; and because $14 + 15 + 2 + 3 = 1 + 4 + 13 + 16 = 34$, if 15 and 2, and 14 and 3 change places, the upper and the lower ranks, and the two middle vertical ranks, will each be 34: and when 5 and 12, and 9 and 8 are shifted in the same manner, we shall have the magic square, *(Fig. 9.)*

Fig. 8.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Fig. 9.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Now with the 16 middle numbers (or from 43 to 58) of the series 1, 2, 3, 4, &c. to 100, make the magic or central square of 16 cells (see *Fig. 10*). Then for the first border take the 10 adjacent numbers on each side of those 16 numbers, and arrange them as below:

33 34 35 36 37 38 39 40 41 42
68 67 66 65 64 63 62 61 60 59

The upper or less numbers being the complements of those below to 101 or $\frac{1}{4}$ of each rank of the square of 6 when completed with the border : hence it follows, that 3 of the greater numbers and 3 of the less, which are not the complements of those 3 greater, must form a rank, and their sum be 303. Now in any arithmetical series the sum of the 3d, 5th, and 6th terms are equal to the sum of the 2d, 4th, and 8th, or the 3d, 4th, and 5th are together equal to the two first with the 9th. Adopting this latter combination, we have $68 + 67 + 60 + 35 + 36 + 37$

for a rank, (because 33 + 34 + 41 = 35 + 36 + 37). Let 68 and 67 be the corner numbers in the column (the lower for example), and put their complements in the opposite corners diagonally; also place the other numbers in the lower column, and their complements in the opposite cells at top; and the upper and lower ranks of the border will be completed.

Fig. 10.

2	3	9	15	97	96	95	94	93	1
12	20	25	26	80	79	78	77	19	89
13	28	34	41	66	65	64	33	73	88
14	29	89	43	57	56	46	62	72	87
16	31	40	54	48	49	51	61	70	85
83	69	59	50	52	53	47	42	32	18
84	71	63	55	45	44	58	38	30	17
90	74	68	60	35	36	37	67	27	11
91	82	76	75	21	22	23	24	81	10
100	98	92	86	4	5	6	7	8	99

To fill the side columns we have 63, 62, 61, and 59, with their complements. First, $303 - (68 + 34) = 201$: now $201 = 63 + 59 + 39 + 40$, which are the numbers for the left column of the border; put their compelments opposite in the other column, and the first border is completed.

The 2d border is constructed with the following 28 numbers, or the 14 next less than 33, and the other 14 next greater than 68.

19 20 21 22 23 24 25 26 27 28 29 30 31 32
82 81 80 79 78 77 76 75 74 73 72 71 70 69

Four of the greater numbers and 4 of the less (their sum being $404 = 4$ pair) are necessary for a column of the border: now the two first terms of an arithmetical series *plus* the 7th and 8th are equal to the sum of the 3d, 4th, 5th, and 6th, or $19 + 20 + 25 + 26 = 21 + 22 + 23 + 24$; therefore the latter numbers with 82, 81, 76 and 75 make 404; put these numbers in the lower column of the border, and let 82 and 81 be the corner numbers, and their complements the other corner numbers diagonally; then fill the opposite cells with the complements of those below.

To fill the left hand column of this border, we have $82 + 20 = 102$ which is greater than 101 by 1, therefore 3 terms of the series

27, 28, 29, 30, 31, 32 must exceed the other three by 1 ; now $28+29+31+1=27+30+32$, or $28+27+32=27+30+31+1$; taking the first 3 will give 28, 29, 31, 74, 71, 69, with their complements, for the left, and right columns.

The remaining 36 numbers for the outward border, when arranged in pairs, will stand as below :

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
100	99	98	97	96	95	94	93	92	91	90	89	88	87	86	85	84	83

Here in the upper, or series of the less numbers, one of the combinations is $1+2+3+9+15=4+5+6+7+8$, and therefore either five, with the complement of the other five, make $505=101 \times 5$ the number in each rank of this border.

Take 4, 5, 6, 7, 8, and 100, 99, 98, 92, 86 for the lowest rank, and make 100 and 99 the corner numbers, then 2 and 1 will be at the other corners; and the other cells at top are filled with the other complements. We now have the 8 numbers 10, 11, 12, 13, 14, 16, 17, 18, with their complements, to fill the two side columns. First, $505 - (100 + 2) = 403$; therefore 4 of those 8 numbers together must exceed the other 4 by 1 : now $12 + 13 + 14 + 16 + 1 = 10 + 11 + 17 + 18$, (or $12+13+14+17=10+11+18+16+1$, &c.) : let 12, 13, 14, 16, and 83, 84, 90, 91, be in the left column, and their complements opposite in the right, and the square is completed ; and the method is similar when the roots are odd numbers.

Hence the construction of these borders depends on the solution of this simple problem.— Having an arithmetical series, to find a proposed number of the terms whose sum shall be equal to the sum of the like number of the other terms : or that the two sums shall have a given difference.

A singular and curious property belongs to these magic squares with borders; it is this :—The sum of the squares of the numbers in any rank of a border, is equal to the sum of the squares of the numbers in the opposite rank of that border : thus in the square *(Fig. 10.)* the sum of the squares of 100, 98, 92, 86, 4, 5, 6, 7, 8, 99 is the same as that of the numbers 2, 3, 9, 15, 97, 96, 95, 94, 93, 1. The numbers, however, are so related, that when the differences of the least number in the lower column and the other numbers in that column are taken from the greatest number in the upper column, the remainders are the other numbers in the upper column : in which case, when the sums are equal, the sums of their squares will also be equal. See an investigation of this property in the answer to Quest. 281, No. 11, Leybourn's Math. Repository.

Hence it appears that an arithmetical series of an even number of terms, which can be formed into a magic square, may be disposed in setts, so that their sums, and also the sums of their squares, shall respectively be equal. Thus *(Fig. 9.)* the sum of the squares of the numbers in the two upper columns, in the two lower, in the two left hand columns, in the two on the right, and also in the two diagonals, are all equal.

A magic square whose root is any number greater than 3 may be constructed by adding a border, or borders. But there are other methods when the roots are *evenly-even*, as 4, 8, 12, 16, &c. Let it be required to make a square of 64 cells with the series 1, 2, 3, 4, &c. to 64.

Arrange the 64 numbers in 4 ranks of pairs, the pairs making alternately 66 and 64, (64 + 2, 63 + 1, 62 + 4, &c.)

			A					C		
2	1		4	3	6	5		8	7
64	63		62	61	60	59		58	57
10	9		12	11	14	13		16	15
56	55		54	53	52	51		50	49

B   D

18	17		20	19	22	21		24	23
48	47		46	45	44	43		42	41
26	25		28	27	30	29		32	31
40	39		38	37	36	35		34	33

G   H

Let the pairs on the left of ABG, and those to the right of CDH be each $= \frac{1}{4}$ of the whole; then the numbers on the left of AB, together with those on the right of CD are to fill one square of 16 cells; those between AB and CD another; the numbers on the left of BG, and on the right of DH, a third; and the middle numbers a 4th.

Now let the upper or less numbers be disposed in 4 series of pairs, thus

1	2	7	8		3	4	5	6		17	18	23	24		19	20	21	22
16	15	10	9		14	13	12	11		32	31	26	25		30	29	28	27

Each pair of the two first divisions being 17, and each of the other two 49. Let those pairs be placed in the alternate halves of the horizontal columns of 4 squares (of 16 cells each); the numbers in the first and third divisions increasing from the middles of the squares to the right and left, but the reverse takes place in the other two. (*Fig.* 11.)

Fig. 11.

If the vacant cells are filled with the corresponding or lower numbers 64, 63, 62, &c. and the 4 squares joined together in the order they stand, the square of 64 cells will be completed. (*Fig.* 12.)

A corresponding number is that which, with the less number, forms a pair: thus 61 must stand above 3; 60 above 6; 64 under 2, &c.

The sum in each of the horizontal, vertical, and diagonal ranks is $= \dfrac{(64+1)\times 32}{8} = 260$.

Fig. 12.

61	52	13	4	45	36	29	20
3	14	51	62	19	30	35	46
60	53	12	5	44	37	28	21
6	11	54	59	22	27	38	43
55	58	7	10	39	42	23	26
9	8	57	56	25	24	41	40
50	63	2	15	34	47	18	31
16	1	64	49	32	17	48	33

The sum in the bent row, or from 50 to 59, and from 31 to 22 (parallel to the diagonal is also $= 260$. And the sum in the next bent row, or from 9 to 5, and from 44 to 40 is also 260, &c.; and the same thing holds, taking the rows on the sides, or upwards, parallel to the diagonals. The sum of the 4 corner numbers of each of the 4 concentric squares is 130 or half a vertical rank. And the sum of every 4 contiguous numbers forming a square is also 130. Some of these properties are noticed by Dr. Franklin, (see his works, vol. 3, edit. 1806). But the Doctor's square of 64 cells is imperfect, because it fails diagonally.

In the same manner we can fill a square of 256 cells ($= 16 \times 16$) thus :

Arrange the 256 numbers (from 1 to 256) in 4 divisions, each division containing two ranks of pairs, the pairs being alternately 258 and 256, or $256+2$, $255+1$, $254+4$, &c. and consequently 4 of the greater pairs, together with 4 of the less, or $258\times4+256\times4 = 2056\left(= \dfrac{(256+1)\times 128}{16}\right)$ the number required in each of the horizontal, vertical, and diagonal ranks of the square :

```
  2   1   4   3   6   5   8   7  10   9  12  11  14  13  16  15
256 255 254 253 252 251 250 249 248 247 246 245 244 243 242 241

 18  17  20  19  22  21  24  23  26  25  28  27  30  29  32  31
240 239 238 237 236 235 234 233 232 231 230 229 228 227 226 225

 34  33  36  35  38  37  40  39  42  41  44  43  46  45  48  47
224 223 222 221 220 219 218 217 216 215 214 213 212 211 210 209

 50  49  52  51  54  53  56  55  58  57  60  59  62  61  64  63
208 207 206 205 204 203 202 201 200 199 198 197 196 195 194 193

 66  65  68  67  70  69  72  71  74  73  76  75  78  77  80  79
192 191 190 189 188 187 186 185 184 183 182 181 180 179 178 177

 82  81  84  83  86  85  88  87  90  89  92  91  94  93  96  95
176 175 174 173 172 171 170 169 168 167 166 165 164 163 162 161

 98  97 100  99 102 101 104 103 106 105 108 107 110 109 112 111
160 159 158 157 156 155 154 153 152 151 150 149 148 147 146 145

114 113 116 115 118 117 120 119 122 121 124 123 126 125 128 127
144 143 142 141 140 139 138 137 136 135 134 133 132 131 130 129
```

Now with these 4 divisions make 4 magic squares of 64 cells each, exactly as the square (*Fig.* 12.) was constructed : these put together will form the magic square of 256 cells.   We shall put down the process for one square with the first division, though it will appear something like a repetition :

```
          A                C
   2    1  |  4    3    6    5  |  8    7
 256  255  | 254  253  252  251 | 250  249

  10    9  | 12   11   14   13  | 16   15
 248  247  | 246  245  244  243 | 242  241
          B                D
  18   17  | 20   19   22   21  | 24   23
 240  239  | 238  237  236  235 | 234  233

  26   25  | 28   27   30   29  | 32   31
 232  231  | 230  229  228  227 | 226  225
          G                H
```

This arrangement is similar to that for the square referred to (*Fig.* 12.)   The upper or less numbers are the same as in that example; therefore the four adjacent squares are copies of *Fig.* 11.

The vacant cells are to be filled with the complements : thus 253 must stand above 3; 243 under 13, &c.   Let the 4 squares be joined in the order they stand, and we shall have a magic square of 64 cells.   This, with three others constructed in the same manner with the 3 remaining divisions, will form the square (*Fig.* 14.) Which is *a magic square of magic squares ;* or a magic square made with other magic squares.

Fig. 13.

		13	4
3	14		
		12	5
6	11		

		29	20
19	30		
		28	21
22	27		

		7	10
9	8		
		2	15
16	1		

		23	26
25	24		
		18	31
32	17		

Fig. 14.

221	212	45	36	205	196	61	52	157	148	109	100	141	132	125	116
35	46	211	222	51	62	195	206	99	110	147	158	115	126	131	142
220	213	44	37	204	197	60	53	156	149	108	101	140	133	124	117
38	43	214	219	54	59	198	203	102	107	150	155	118	123	134	139
215	218	39	42	199	202	55	58	151	154	163	106	135	138	119	122
41	40	217	216	57	56	201	200	105	104	153	152	121	120	137	136
210	223	34	47	194	207	50	63	146	159	98	111	130	143	114	127
48	33	224	209	64	49	208	193	112	97	160	145	128	115	144	129
253	244	13	4	237	228	29	20	189	180	77	68	173	164	93	84
3	14	243	254	19	30	227	238	67	78	179	190	83	94	163	174
252	245	12	5	236	229	28	21	188	181	76	69	172	165	92	85
6	11	246	251	22	27	230	235	70	75	182	187	86	91	166	171
247	250	7	10	231	234	23	26	183	186	71	74	167	170	87	90
9	8	249	248	25	24	233	232	73	72	185	184	89	88	169	168
242	255	2	15	226	239	18	31	178	191	66	79	162	175	82	95
16	1	256	241	32	17	240	225	80	65	192	177	96	81	176	161

The horizontal, vertical, and diagonal ranks of the 4 squares are each 1028, or just half a rank of the great square. And if we take the numbers in any bent row of the great square parallel to the diagonals, either downwards, upwards, or sideways, their sum is 2056 : thus from 247 to 200, and from 105 to 90 the sum is 2056, &c. Also the 4 corner numbers of each of the 8 concentric squares make 514 or ¼ of a rank : thus 46 + 131 + 82 + 255 (at the corners of the 7th square) make 514.—If the 4 outward columns all round are taken away, there will remain a magic square of 64 cells. And the sum of the numbers in every 16 contiguous cells forming a square is 2056 ; in other words, if a square hole be cut in a paper just the size of 16 cells, and the paper laid any where upon the square, the numbers appearing through the hole will together make 2056. Dr. Franklin's square of 256 cells answers in this respect : but here again he has failed in an essential property of a magic square ; his diagonals, instead of being equal, are 1297 and 2184 ; this inequality seems to have been the consequence of his not having followed a regular alternation in filling the half columns *.

---

* The engraver has made three mistakes in the Doctor's square ; 181 in the corner should be 185 ; 241 in the 7th column should be 211 ; and the figure 3 is wanting in the number 203 in col. 1.

Thus we have a square much more magical than the Doctor's famous square, which he sets down as " the most magically magical of any magic square ever made by any magician."

It will not be difficult to comprehend the *rationale* of this mode of construction. The sum of each two numbers in the two squares to the left (*Fig.* 13.) is 17, and in the other two 49, therefore the sums of their complements are respectively equal; and consequently when the latter are placed in the vacant cells, each horizontal and vertical rank will be 17 + 497 = 514 in the two squares to the left; and 49 + 465 = 514 in the other two (497 and 465 being the respective sums of the complemental numbers): the diagonals, however, are 498 and 530, but 498 + 530 = 1028, therefore in joining the 4 squares (*Fig.* 13.) together, the diagonal 498 of one square, and that which is 530 of another, must make a diagonal of the square of 64 cells. And in putting together the 4 squares to form the large square, it is necessary that the sum of each 5th pair of numbers (counting from the right, or from the left, or from the top downwards) where the halves of the square join, should respectively be the same: thus 129 + 84 = 145 + 68 = 193 + 20, &c. or 52 + 157 = 58 + 151, &c. the property respecting every 16 cells evidently depends on this:

If the square be cut horizontally, or vertically, through the middle, the halves may change places, and the properties of the square will remain as before.

The construction of a common magic square of magic squares is more simple than the foregoing. Thus let the series be 1, 2, 3, 4, &c. to 64. With the 8 first numbers of the series and the 8 last, make a magic square of 16 cells (*Fig.* 15.) according to the method for *Fig.* 9. And with the eight numbers next greater than 8, and the 8 numbers next less than 57, fill another in the same manner (*Fig.* 16.) The 8 numbers next greater than 16, and the 8 numbers next less than 49 are for the 3d square; and the remaining 16 middle numbers for the 4th.

The columns and diagonals of these 4 squares are 130 each, and consequently they may be joined together in any order so as to form a square. This is the most expeditious method of constructing magic squares whose roots are multiples of 4.

To make a square of this kind when the root is 9 (*Fig.* 17.) Let the series be 1, 2, 3, 4, &c. to 81. Divide this series into 9 other series of 9 numbers each, as follow :

Fig. 15.

1	63	62	4
60	6	7	57
8	58	59	5
61	3	2	64

17	47	46	20
44	22	23	41
24	42	43	21
45	19	18	48

Fig. 16.

9	55	54	12
52	14	15	49
16	50	51	13
53	11	10	56

25	39	38	28
36	30	31	33
32	34	35	29
37	27	26	40

```
1  10  19  28  37  46  55  64  73
2  11  20  29  38  47  56  65  74
3  12  21  30  39  48  57  66  75
              &c.
```

If a magic square be made with each of these series, the ranks in the squares will respectively be 111, 114, 117, 120, 123, 126, 129, 132, 135; and a magic square constructed with these numbers would have

$$\frac{(135 + 111)}{3}\ 4\tfrac{1}{2} = 369\ \text{in}$$

each rank, the same as the square (*Fig.* 4.), the 9 squares must therefore be

Fig. 17.

31	76	13	36	81	18	29	74	11
22	40	58	27	45	63	20	38	56
67	4	49	72	9	54	65	2	47
30	75	12	32	77	14	34	79	16
21	39	57	23	41	59	25	43	61
66	8	48	68	5	50	70	7	52
35	80	17	28	73	10	33	78	15
26	44	62	19	37	55	24	42	60
71	8	53	64	1	46	69	6	51

arranged in the same order as the figures stand in *Fig.* 3, or in the squares themselves.

Four squares of 25 cells each will make the square of 10; and 4 squares of 49 cells, the square of 14, &c.

If a magic square is constructed with a geometrical progression, the *products*, instead of the sums, will be equal.

There are other methods of constructing these squares; for (the square of 9 cells excepted) they all admit of many changes. In those squares of the most simple construction, such as *Fig.* 2, 4, and 9, we get varieties by merely transposing the columns that are equally distant from the centres of the squares. Thus in *Fig.* 2, columns 3 and 4, may take place of each other: columns 1 and 2, and 4 and 5 may be shifted to the opposite sides; or the two first, and the two last, may respectively change places and remain on the same sides of the squares, &c. In this manner any number in the square (*Fig.* 9.) may be shifted to any proposed cell in the square. And hence it is evident that the permutations which may take place, particularly in the larger squares, must be extremely numerous; but the computations for the exact number in any case are very tedious.

The earliest European authors whose works contain any thing upon magic squares, are Cor. Agrippa, Stifelius, and Bachet. Afterwards came Frenicle, who wrote professedly on magic squares, and first constructed those with borders. In 1704, M. Poignard published a treatise on the subject, in which is introduced a new species of magic squares, made with progressions repeated as often as there are units in the roots. *Fig.* 18, is a square of this kind, constructed with the series 1 2 3 4 taken 4 times, which are so disposed that neither

Fig. 18.

2	1	3	4
3	4	2	1
4	3	1	2
1	2	4	3

of the numbers occur twice in any rank. By sub-stituting letters, however, it appears that the num-bers for this purpose are not necessarily confined to progressions of any kind. Figures 19, 20, 21, are three squares filled with letters repeated; but the same letter is not found twice in any column or diagonal. Now it is evident that the letters may represent any number whatever.

Fig. 19.

a	b	c	d
d	c	b	a
b	a	d	c
c	d	a	b

Fig. 20.

a	b	c	d	f
b	d	f	c	a
f	c	b	a	d
c	a	d	f	b
d	f	a	b	c

Fig. 21.

a	b	c	d	f	g
g	f	d	c	b	a
b	c	g	a	d	f
d	a	f	b	g	c
f	d	a	g	c	b
c	g	b	f	a	d

Ozanam, in his Math. Recreations; and M. M. De la Hire, and Sauveur, in the Mem. of the French Acad. 1705, 1710, have also treated the subject of magic squares at considerable length.

Among the ancients magic squares were held in great veneration, and supposed to be endowed with occult virtues; and it appears that they still are used as talismans in the East. A magic square of 16 cells stands over the door of a house in Bengal, and underneath is the following prayer or supplication in the Persian language :

O God preserve the Doctor of the Faith, surnamed Karkhy, from the calamities of this world. May he be always a favorite of heaven, whilst Moses is selected as porter to Aly.

IV. QUESTION 62, *by Mr.* William Whitworth, *of Northampton.*

Required the dimensions of a leaden cistern, the form a paral-lelopipedon, to hold just 50 bushels, the length and breadth to be in the ratio of 5 to 3, the lead half an inch thick and the least quan-tity possible; also, what the cistern cost at $1\frac{1}{4}d.$ per pound ?

Answered by Anna Philomathes.

Let $b = 5$, $c = 3$, and $aaa = 107520 = $ cubic inches in 50 bushels, also let $x = $ length: then will $\dfrac{cx}{b} = $ breadth, $\dfrac{baaa}{cxx} = $ to the depth, and $\dfrac{ccxxx + 2bbaaa + 2bcaaa}{bcx} = $ to least superficies of cistern.

The fluxion thereof is $\dfrac{2ccx^3\dot{x} - 2bba^3\dot{x} - 2bca^3\dot{x}}{bcxx} = 2ccx^3\dot{x} - 2bba^3\dot{x}$

$-2bca^3\dot{x} = 0.$ Therefore $x = \sqrt[3]{\dfrac{bba^3 + bca^3}{cc}} = \text{length} = 78\cdot181426$

inches, $\dfrac{cx}{b} = \text{breadth} = 46\cdot9088$ inches, and $\dfrac{ba^3}{cx} = \text{depth} =$

$29\cdot318$ inches.

The purchase $14l.$ $1s.$ $7\frac{3}{4}d.$

Additional Solution.

The above original solution is false as well as that given in the *Diarian Repository*, they both making the internal surface a *minimum*, which is contrary to the intention of the question; for it expressly declares " To have it that the least lead will require," and consequently the difference between the solids made up of the external and internal dimensions must be the *minimum*. Wherefore putting the

$$
\text{internal}
\begin{cases}
\text{length} = 5x \\
\text{breadth} = 3x \\
\text{depth} = y
\end{cases},
\quad
\begin{array}{c}
\text{the external} \\
\text{dimensions} \\
\text{will be}
\end{array}
\begin{cases}
5x + 2 = \text{length} \\
3x + 1 = \text{breadth} \\
y + \frac{1}{2} = \text{depth}
\end{cases}.
$$

And hence $15x^2y = 107520$, and $(5x + 1).(3x + 1).(y + \frac{1}{2}) -$ $15xy$ or $\frac{15}{2}x^2 + (8x + 1).(y + \frac{1}{2}) = $ a *minimum*. The value of y being exterminated out of this equation, by means of the first, and its fluxion put $= 0$, we at last obtain $15x^4 + 4x^3 - 57344x - 14336 = 0$; in which equation the root x is $= 15\cdot630773$. Wherefore $5x = 78\cdot135865 = $ internal length, $3x = 46\cdot892319 = $ internal breadth, and $y = \dfrac{107520}{15xx} = \dfrac{7168}{xx} = 29\cdot338444 = $ internal depth; also by adding 1 to each of the former numbers and $\frac{1}{2}$ to the latter, we have $79\cdot153865$, $47\cdot892319$, and $29\cdot838444$ for the external length, depth, and breadth. From the continual product of these three taking 107520 the content of the inside, there remains $5594\cdot34$ for the cubic inches of lead in the cistern; and this multiplied by $\cdot409618lb.$ the weight of a cubic inch of lead according to *Ward*, the product is $2291\cdot54lb.$ of lead, which at $1\frac{1}{2}d.$ a pound, amounts to $14l.$ $6s.$ $5d.$ $1\cdot24q.$ H.

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v.   QUESTION 63, *by Mr. Lewis Evan, from Carmarthenshire.*

Ten persons (A, B, ... K) purchase the hay to be produced upon a certain meadow for $40l.$ The first contributes a certain sum, the second $5s.$ more than the first, the third $5s.$ more than the second, and so on to the last; the form of the meadow is that of a right-angled triangle, the hypothenuse being 50, and the other two sides 30 and 40 chains. They agreed to have the meadow divided into shares, by lines perpendicular to the hypothenuse or longest side,

which should be proportional to the sums contributed. The share of the first to be adjacent to the less acute angle, that of the second adjacent to that of the first, &c. How must the hypothenuse be divided?

Answered by Anna Philomathes.

Sharers	Must pay			Share of			Len. on base.	Length perp.
	l.	*s.*	*d.*	*a.*	*r.*	*p.*	*chains.*	*chains.*
A	2	17	6	4	1	10	10·723	8 04
B	3	2	6	4	2	30	4·768	11·62
C	3	7	6	5	0	10	3·873	14 52
D	3	12	6	5	1	30	3 438	17·10 ·
E	3	17	6	5	3	10	3·177	19·48
F	4	2	6	6	0	30	3·002	21 73
G	4	7	6	6	2	10	2·876	23·89
H	4	12	6	6	3	30	3·140	20·00
I	4	17	6	7	1	10	4·261	14·32
K	5	2	6	7	2	30	10·738	*a* △
Sum	40	0	0	60	0	0	49·999	

Solution.

By the nature of arithmetical progression, having given the common difference $5 = d$, the number of terms $10 = n$, and the sum of the series $40 \times 20 = 800 = s$; there will be found the first term

$$a = \frac{2s - dn \cdot (n-1)}{2n} = \frac{1600 - 450}{20} = 57\tfrac{1}{2}s. = 2l.\ 17s.\ 6d. =$$

A's share of the money, and by the constant addition of 5s. all the rest will be found to be as in the second column of the original solution above.

Then, their shares of the field will be in proportion as their sums of money, and since the triangle is evidently right-angled, and its area $= 30 \times 20 = 600$ chains $= 60$ acres, we shall have, as 40l. : 60 acres or as 2 : 3 :: 2l. 17s. 6d. : 4·3125 acres $= 4ac.$ 1ro. 10po. $=$ A's share, and as 2l. : 3 :: 5s. : ·375ac. $=$ 1ro. 20po. $=$ the common difference; which being continually added will give the several shares as in the 3d column.

Again, for the bases and perpendiculars: Since the first and last

parts are triangles similar to the whole field, and similar triangles being as the squares of their like sides, we shall have,

$$\text{as } \sqrt{60} : \sqrt{4\cdot3125} :: \begin{cases} 40 : 10\cdot723 = \text{A's base} \\ 30 : 8\cdot042 = \text{A's perpendicular,} \end{cases}$$

$$\text{and } \sqrt{60} : \sqrt{7\cdot6875} :: \begin{cases} 40 : 14\cdot317 = \text{x's perpendicular} \\ 30 : 10\cdot738 = \text{x's base.} \end{cases}$$

Then to the first and last areas add their adjacent ones, and the sums will be the next similar triangles; whose sides being calculated in the same manner, their perpendiculars will be those in the last column, and the differences of the bases will be the bases in the column next to it.　H.

VI.　QUESTION 64, *by Mr.* Rob. Beales,

A person has a legacy of 200*l.* left him, but which is not to be paid till the end of $9\frac{1}{4}$ years hence; being in want of money, he agrees to sell it for 100*l.* to be paid immediately: What rate per cent. per annum compound interest does the purchaser obtain for his ready money?

Answered by Anna Philomathes.

Seven pounds seven shillings and fourpence farthing per cent. per annum, compound interest.

Solution.

If r be put for rate of 1*l.* by the nature of compound interest, we shall have $100 \times (1 + r)^{\frac{39}{4}} = 200$, or $(1 + r)^{\frac{39}{4}} = 2$, or $(1 + r)^{39} = 2^4 = 16$; and $r = 16^{\frac{1}{39}} - 1 = \cdot07368$. Hence the rate per cent. is $100 \times \cdot07368 = 7\cdot368 = 7l.\ 7s.\ 4d.\ 1\cdot28q.$　H.

PRIZE QUESTION.

Required the sides and radii of the circumscribing circles of two regular polygons, the less of five sides and the other of seven, from the following data, *viz.* the less polygon is to the greater as 3 to 13, and the line subtending the angle formed by two adjacent sides of the pentagon is equal to the length of a pendulum which vibrates 61 times in a minute.

Answered by Mr. W. Whitworth.

The lengths of pendulums are reciprocally proportionable to the squares of their vibrations.

Therefore as square $61 = 3721 : 3600 :: 39{\cdot}2 : 37{\cdot}9252 = b$ the subtense, whence the side a must be found by this theorem: If a quadrangle be inscribed in a circle, the rectangles of the two diagonals will be equal to both the rectangles of the opposite sides.

Now the diagonals being each equal to the subtense b, the theorem is $aa + ba = bb$; ergo, $a = \sqrt{(bb + \frac{1}{4}bb)} - \frac{1}{2}b = 23{\cdot}439062$ inches, the side of the pentagon.

Then b being the hypothenuse of a rectangle triangle described in a circle, and cd part of the diameter (as far as where the perpendicular falls) the base, the perpendicular being $\frac{a}{2}$; we have $\sqrt{\left(bb - \dfrac{aa}{4}\right)}$ $= cd$; and $\frac{1}{4}aa \div cd = df$ the other part of the diameter; and $cd + df =$ diameter required. Also the area of the pentagon will be $945{\cdot}210782$. Then $3 : 13 :: 945{\cdot}2107 : 9095{\cdot}9133912 =$ area of heptagon, $3{\cdot}633912 : 1 :: 4095{\cdot}91$ in duplicate ratio, to side of the heptagon. Then the other parts are easily found, as below.

Parts.	Pentagon.	Heptagon.
The area of the	945·2107	4095·9133
Diameter of circumscribing circle	39·8768	76·8283
Diameter of inscribed circle	32·2611	69·1047
Circumscribing circle	125·2768	241·3637
Inscribed circle	101·3510	217·0994

—••••••—

Questions proposed in 1719, *and answered in* 1720.

I. QUESTION 65, *by Mr. Henry Walker.*

There is a piece of gold in the form of a parallelopipedon, and the product of the length and depth in inches is 96; the product of the breadth and depth is 80, and the product of the length and breadth 120; what are the dimensions themselves, and what is the weight and value of the piece?

Answer.

The length 12, the breadth 10, and depth 8 inches, and if $\cdot10083$ inch $= 1$ ounce, and the troy pound $= 42l.\ 10s.$ the value was $337l.\ 19s.\ 9\frac{1}{2}d.$

Solution.

From the question it is evident that the breadth is to the length as 80 to 96, or as 5 to 6, and to the depth as 120 to 96 or as 5 to 4.

Now the product of 4 and 5 is 20, which is similar to the product 80 in the question; and each of the proportional numbers 4, 5, 6 will be to the corresponding numbers required, as the root of 20 to the root of 80. Wherefore as $(\sqrt{20} : \sqrt{80} :: \sqrt{1} : \sqrt{4} ::)$

$$1 \therefore 2 :: \begin{cases} 4 : 8 = \text{the depth,} \\ 5 : 10 = \text{the breadth,} \\ 6 : 12 = \text{the length.} \end{cases}$$

Hence the content is $8 \times 10 \times 12 = 960$ inches. And, using Ward's tables, $960 \times 10\cdot359273$ ounces $= 80 \times 10\cdot359273 = 828\cdot74184$lb. which at 42l. 10s. per lb. amount to 35221l. 10s. 6d. 3·072q. H.

Algebraically.

Let x, y, and z represent the length, breadth, and depth of the wedge; then, by the question, $xz = 96$, $yz = 80$, and $xy = 120$. The product of these equations is $x^2y^2z^2 = 921600$, the square root of which is $xyz = 960$. Divide this equation by each of the given ones and we have $x = 12$, $y = 10$, and $z = 8$, as above. L.

II. QUESTION 66, by Mr. W. Doidge.

In the latitude of 50° 48′ a beautiful rainbow was observed bearing N. E. ¼ E. What was the time of the day?

Answered by Mr. William Armstrong.

If the sun was near the equinox, which is unlimited in the question, it was 33 min. 45 seconds past 3 evening.

Solution.

In the above original solution to this question, it is justly remarked that the question is unlimited, as the declination or time of the year should have been given, and then the solution would have been very easy: For as the rainbow is formed directly opposite to the sun with regard to the observer, the bearing of the sun will be S. W. ¼ W. so that in a spherical triangle would be given two sides (= the colat. and codeclin.) and an angle (the sun's azimuth or bearing from the north) opposite to one of them (the codeclin.), to find their included angle = the measure of the time from noon required. H.

III. QUESTION 67, *by Mr.* W. Hawney.

How many different ways is it possible to pay one hundred pounds with guineas and pistoles only, reckoning the guineas at 21s. 6d. and the pistoles at 17s. each.

Answered by Mrs. Hesychia.

Three; *viz.* Guineas 10, 44, 78
           Pistoles 105, 62, 19.

Solution.

Put x and y for the number of guineas and pistoles respectively, then the number of sixpences in them being $43x$ and $34y$, and the sixpences in 100l. $= 4000$, we shall have $43x + 34y = 4000$;

hence $x = \dfrac{4000 - 34y}{43} = 93 - \dfrac{34y - 1}{43} = $ a whole number;

consequently $\dfrac{34y - 1}{43}$ as also 5 times the same, *viz.* $\dfrac{170y - 5}{43} = $ a

whole number; take this from the whole number $\dfrac{172y}{43} = 4y$, and

there remains the whole number $\dfrac{2y + 5}{43}$; from 22 times this whole

number, *viz.* $\dfrac{44y + 110}{43}$, take the whole number $\dfrac{43y}{43} = y$, and

there remains the whole number $\dfrac{y + 110}{43}$, which put $= p$; then y

$= 43p - 110$, where the least value of p is evidently 3. Substitute this

value of y in the value of x, and we shall have $x = \dfrac{4000 - 34y}{43} =$

$180 - 34p$, in which the greatest value of p is evidently 5. So that the question admits of only three solutions, p having only three values, *viz.* 3, 4, and 5; and by writing each of these numbers for p in the values of x and y, we obtain $x = 78$, 44, and 10, and $y = 19$, 62, and 105. H.

IV. QUESTION 68, *by Mr.* John Leddell.

There is a segment of a circle the chord of which is 60 feet, and versed sine 10 feet; what will be the versed sine of that segment of the same circle whose chord is 90 feet?

Answer.

The versed sine's length is 28·2055 feet.

Solution.

The right sine being 30, and versed sine 10, by the nature of the circle, the diameter will be $10 + 30^2 \div 10 = 100$, and consequently the radius $= 50$. But, by right-angled triangles, the cosine of the arc whose right sine is 45, will be $= \sqrt{(50^2 - 45^2)} = 5\sqrt{(10^2 - 9^2)} = 5\sqrt{19} = 21 \cdot 7944947$, &c. which being taken from the radius 50, there remains $28 \cdot 205505$, &c. for the versed sine required. **H.**

V. , QUESTION 69, by Mr. J. Smith.

Three persons A, E, and Y, bought a piece of land in the form of an ellipse, for 180l. The conjugate axis is 40 chains and the distance of the focus from the centre 15 chains, the abscissa of Y's part is to be 18 chains, and the rectangular ordinate of E's part 15 chains; A's share lies between those of Y and E, what is each person's share of the land, and what sum ought each to pay?

Answer.

A's part 96·65864 pounds.
E's .... 25·19722
Y's .... 58·14413.

Solution,

By the right-angled triangles, the distance between the focus and the end of the conjugate will be $\sqrt{(20^2 + 15^2)} = 25 =$ the semi-transverse by the nature of the ellipse; then, by case 2, p. 225 Mensuration, $25 - \dfrac{25\sqrt{(20^2 - 15^2)}}{20} = 25 - \dfrac{25\sqrt{(4^2 - 3^2)}}{4} = 25 - \dfrac{25\sqrt{7}}{4} = 8 \cdot 4640543$, &c. $=$ the abscissa of E's part.

Now $\frac{8 \cdot 4640543}{50} = \cdot 169281$, and $\frac{18}{50} = \cdot 36$; but the tabular areas for these two quotients, in the table of segments at the end of my Mensuration, are ·08799586 and ·25455055, whose sum taken from ·78539816, there remains ·44285175. Also the product of the two axes $= 50 \times 40 = 2000$.

Then, by rule 2, p. 244 Mensuration, we shall have
·08799586 × 2000 = 175·99172 = E's part ⎫
·25455055 × 2000 = 509·10110 = Y's part ⎬ in square chains;
·44285175 × 2000 = 885·70350 = A's part ⎭

their sum being $1570 \cdot 79632 = \cdot 78539816 \times 2000 =$ the area of the whole ellipse.

Also, by proportion, we have as 1570·79632 : 180l.

$$:: \begin{cases} 175\cdot99172 : & 20\cdot16721. = 20l. \ 3s. \ 4d. \ 0\cdot512q. \ = \text{E's money}, \\ 509\cdot10110 : & 58\cdot3387 \ = \ 58 \quad 6 \quad 9 \ 1\cdot152 \quad = \text{Y's}, \\ 885\cdot70350 : & 101\cdot4941 \ = 101 \quad 9 \ 10 \ 2\cdot336 \quad = \text{A's}. \end{cases}$$

                                   180 0 0 0   sum.

So that the numbers in the original solution are false.   H.

IV. QUESTION 70, *by Mr. J.* Dodd.

To find the area of a triangular piece of land there is given one of the angles 30° 43′, and the rectangle of the including sides 168·75 chains.

Answered by Mr. John Finch, *of Norwich.*

As rad. : s. given angle :: ¼ the rectangle given : area 4a. 1r. 9p.

Solution.

By a known rule for the area of a triangle, we have as 1 : ·510793 (the nat. sine of 30° 43′) :: 84·375 = half the product of the including sides : 43·09817 chains = 4a. 1r. 9·5707 perches.   H.

VII. QUESTION 71, *by Mr.* Alex. Naughley, *of Cumberland.*

Required the dimensions of a stone trough to contain 5400 cubic inches ; and the sum of its length, breadth, and depth to be 57 inches ?

Answered by Mr. Nath. Brown, *Writing-master, at Sleeford, in Lincolnshire.*

$$\begin{matrix} \text{Length} & 30 \\ \text{Breadth} & 15 \\ \text{Depth} & 12 \end{matrix} \Big\} \text{ inches without } \begin{Bmatrix} 23\cdot777 \\ 11\cdot888 \\ 9\cdot511 \end{Bmatrix} \text{ within.}$$

Solution.

Put x, y, z for the length, breadth, and depth of the stone respectively : Then, by the question, $x + y + z = 57$, and $xyz = 5400$. The question then is unlimited, there being one equation less than the number of unknown quantities. We shall therefore assume another condition, *viz.* we shall suppose the length to be double the breadth, or $x = 2y$: Then, this written for it in the two given equations, we have $3y + z = 57$, and $2y^2z = 5400$.

Hence $z = 57 - 3y$, which written for it in the last equation,

we have $114y^2 - 6y^3 = 5400$, or $19y^2 - y^3 = 900$.   Here $y = 15$.
Consequently $x = 2y = 30$, and $z = 57 - 3y = 57 - 45 = 12$.

Then, since 5 pecks $= 10 \times 268 \cdot 8$ inches, and similar solids are as the cubes of their like sides, we shall have as $\sqrt[3]{5400} : \sqrt[3]{2688}$ or as $\sqrt[3]{225} : \sqrt[3]{112}$ or as $\sqrt[3]{3375} : \sqrt[3]{1680}$, that is, as

$$15 : 2\sqrt[3]{210} :: \left\{ \begin{array}{l} 30 : 4\sqrt[3]{210} = 23 \cdot 77568 \\ 15 : 2\sqrt[3]{210} = 11 \cdot 88784 \\ 12 : \frac{8}{5}\sqrt[3]{210} = 9 \cdot 51027 \end{array} \right\} = \text{the dimensions}$$

of the hollow.   H.

VIII.   QUESTION 72, *by Mrs.* Mary Nelson.

A prize was divided by a captain among his crew in the following manner: the first man took 1l. and one hundredth part of the remainder; the second 2l. and one hundredth part of the remainder; the third 3l. and one hundredth part of the remainder; and they proceeded in this manner to the last, who took all that was left, and it was then found that the prize had by this means been equally divided amongst the crew.   Now if the number of men of which the crew consisted be added to the number of pounds in each share, the square of that sum will be equal to four times the number of pounds in the chest: How many men did the crew consist of, and what was the share of each?

Answer.

The men were 99, and had 99l. each.   The whole 9801l.

Solution.

Put x for the number of pounds in the prize: Then, by the question, $1 + \dfrac{x-1}{100} = \dfrac{99+x}{100} =$ the first man's share, and consequently the sum afterwards left $= x - \dfrac{99+x}{100} = \dfrac{99x - 99}{100}$;

again, the second man's share will be $2 + \dfrac{\dfrac{99x-99}{100} - 2}{100} =$

$\dfrac{198 + \dfrac{99x-99}{100}}{100} = \dfrac{19701 + 99x}{10000}$, which must be $=$ that of the former, that is, $\dfrac{99+x}{100} = \dfrac{19701 + 99x}{10000}$; hence $9900 + 100x =$

19701 + 99x, and $x =$ 19701 — 9900 = 9801 = the number of

pounds in the prize. And $\dfrac{99 + x}{100} =$ 99 = each man's share.

Again, putting $z =$ the number of men, by the question it will be $(z + 99)^2 =$ 4 × 9801; hence, by extracting the root, $z + 99 =$ 2 × 99, and $z =$ 99 = the number of men the same with the number of pounds in each man's share. н.

IX. QUESTION 73.

Find the least number which will divide by given divisors without leaving a remainder : and give an example by finding the least number which will divide by the nine digits without leaving a remainder ?

Answered by Mr. Thomas Fletcher.

Two thousand five hundred and twenty.

Mr. *Naughley* gives this canon ; multiply the highest given powers together, and their products continually by every such given prime, as is no component of any of the said powers, or their product, or of any other given prime : So 9 × 8 × 7 × 5 = 2520 required.

Solution.

The general rule for this kind of questions, is to begin with the greatest number and descend gradually to the least, taking the continual product of them ; but omitting such of the less numbers or of their factors as are factors of any of the greater ones. So, in this example, 9 must be used as being the highest number ; 8 must be used, because neither itself nor any of its factors 2, and 4, are factors of 9 ; 7 must be used because it is no factor either of 9 or 8 ; 6 must be omitted because one (3) of its factors divides 9, and the other (2) divides 8 ; 5 must be used as being no divisor of either 7, 8, or 9 ; but 4 must be omitted as being a factor of 8, 3 must be omitted as being a factor of 9, and 2 omitted as being a factor of 8. The reason of all which is extremely obvious.

Then 9 × 8 × 7 × 5 = 2520 = the least number required. н.

X. QUESTION 74, *by Mr.* Crabb.

The dimensions of a spheroidical cask are as follows ; head diameter 31·8 inches ; bung diameter 40 inches, and length 52·8 inches : What is the difference between its greatest inscribed cylinder and cube in wine gallons ?

Resolved by Mr. T. Hill.

The content of the cylinder 182·2503, of the cube 84·2645 ; the difference in wine gallons 97·9858.

Solution.

Let CB be half the length of the cask, CE and CG half the height of the cylinder and cube respectively ; draw the ordinates EF, GI, and to the point F draw a tangent meeting the semi-transverse CA, and semi-conjugate CD, produced, in T and K.
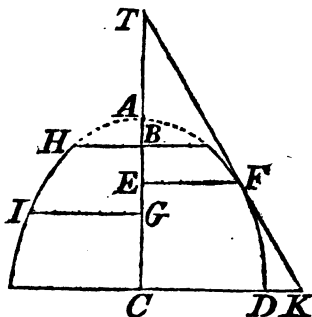
Then, by p. 209 Simpson's Geom. 2d edit. the sub-tangent is twice the altitude of the greatest cylinder, that is, ET $=$ 2EC ; hence CT $=$ 3CE, and CK $=$ $\frac{3}{2}$EF : but, by the nature of

the ellipse, $\left\{ \begin{array}{l} \text{CE : CA :: CA : CT or 3CE} \\ \text{EF : CD :: CD : CK or } \frac{3}{2}\text{EF} \end{array} \right\}$ hence CE $=$ CA$\sqrt{\frac{1}{3}}$, and

EF^2 $=$ $\frac{2}{3}$CD^2 ; but, also by the ellipse, $\sqrt{(\text{CD}^2 - \text{BH}^2)}$: CB :: CD : CA

$= \dfrac{\text{CD} \times \text{CB}}{\sqrt{(\text{CD}^2 - \text{BH}^2)}}$; and therefore CE (or CA$\sqrt{\frac{1}{3}}$) $= \dfrac{\text{CD} \times \text{CB}}{\sqrt{(3\text{CD}^2 - 3\text{BH}^2)}}$.

Hence the solidity of the whole greatest cylinder will be 2CE $\times$ EF^2 $\times$

3·14159, &c. $= \dfrac{4\text{CD}^3 \times \text{CB}}{\sqrt{(3\text{CD}^2 - 3\text{BH}^2)}} \times \dfrac{3\cdot14159,\&\text{c.}}{3} = \dfrac{20^3 \times 26\cdot4 \times 4}{\sqrt{(3\cdot20^2 - 3\cdot15\cdot9^2)}}$

$\times \dfrac{3\cdot14159, \&\text{c.}}{3} = 42100\cdot05.$

Again, CG being half the side of the cube, and GI the radius of a circle circumscribing one face of it, the square of the side of the cube or 4CG^2 will, by right-angled triangles, be $=$ 2IG^2 ; and hence IG^2 $=$ 2CG^2 : but, by the ellipse, CA^2 : CD^2 :: CA^2 — CG^2 : GI^2 or 2CG^2 ; hence

$\sqrt{(2\text{CA}^2 + \text{CD}^2)}$: CD :: CA : CG $= \dfrac{\text{CA} \times \text{CD}}{\sqrt{(2\text{CA}^2 + \text{CD}^2)}} =$ (by writing

the value of CA instead of it) $\dfrac{\text{CD} \times \text{CB}}{\sqrt{(\text{CD}^2 + 2\text{CB}^2 - \text{BH}^2)}}$; whose cube is

$\dfrac{20^3 \times 26\cdot4^3}{(20^2 + 2\cdot26\cdot4^2 - 15\cdot9^2)^{\frac{3}{2}}} = 19464\cdot41 =$ the content of the cube.

The difference of these two contents is 22635·64 inches $=$ 97·9898 wine gallons $=$ the difference required.   H.

PRIZE QUESTION.

There is a wheel nave in the form of a frustum of a spheroid, in

which it is required to make a hole for the axletree, in the form of
the frustum of a cone, capable of holding a gallon of corn; the di-
mensions of the hole to be as follows: the length of the hole which is
also the length of the nave 18 inches, and the ratio of the diameters
of its ends to be that of 13 to 16. The diameter of the nave in the
middle is four times that of the greater end of the hole, and the di-
ameter of the nave at the end is three times the diameter of the greater
end of the hole. Required the solidity of the nave, also required the
periphery of the two hoops, at the distance of four inches from the
ends?

Answered by Mr. Whitworth.

Put $b = 13$, $c = 16$, $l = 18$ the length, $d = 269$ the content,
$g = \cdot78539$ &c. $a =$ lesser base's diam. Then $b : c :: a : ca \div b$
$=$ greater base's diam. Now the area of the greater base, lesser

base, and a geometrical mean multiplied by $\dfrac{l}{3} = d$, the content, that

is, $\left(aag + \dfrac{ccaag}{bb} + \dfrac{aagc}{b}\right) \times \dfrac{l}{3} = d$, which reduced is $ccaa +$

$bbaa + bcaa = 3dbb \div gl$: by this equation $a = 3\cdot903886$ the
lesser base of the cone's frustum, and $ca \div b = 4\cdot804782$ the
greater base's diameter; whence the greatest thickness of the nave
is .... $19\cdot219128$
   Each end .... $14\cdot414346$
   The solidity of the frust. the cavity subt. .... $4191\cdot408136$
   The diameter 4 inches from the end .... $17\cdot874505$
   Circumference .... $56\cdot154402$

Additional Solution.

Putting $16x$ and $13x$ for the diameters of the ends of the conic frus-
tum, and $n = \cdot785398$ &c. by cor. 6, p. 156 Mensuration, we shall

have the content of it $= \left((16 + 13)^2 - 16.13\right) \times x^2 \times \dfrac{18n}{3} = 269$

by the question; hence $x = \sqrt{\dfrac{269}{(29^2 - 16.13) \times 6n}} = \sqrt{\dfrac{269}{633 \times 6n}} =$
$\cdot3002989$. Therefore $16x$ and $13x$ become $4\cdot8047824$ and $3\cdot9038857$
for the diameters of the hole.

The greater diameter drawn into 4 and 3 give $19\cdot2191296$ and
$14\cdot4143472$ for the diameter at the middle and ends of the nave; and
consequently, its solidity, including the hole, will be

$(14\cdot4143472^2 + 2.19\cdot2191296^2) \times \dfrac{18n}{3} = 4460\cdot386$; from which

taking 269 the content of the hole, there remains 4191·386 for the quantity of the solid.

Also, by the nature of the ellipse (see the last fig.) $CB^2 : CG^2 ::$ $CD^2 - BH^2 : CD^2 - GI^2$; and hence $2GI =$

$$2 \sqrt{\frac{CB^2 \cdot CD^2 + CG^2 \cdot BH^2 - CG^2 \cdot CD^2}{CB^2}} = 17·8745048 \text{ (CB being 9 and}$$

$CG = 5$ by the question); which multiplied by 3·14159 &c. we have 56·15441 for the circumference of the hoop.   H.

Questions proposed in 1720, and answered in 1721.

I. QUESTION 75, by Mr. Tho. Dodd, April 30, 1720.

A person had 320l. due last Michaelmas, and five years from that date he will have 202l. more due; now he would be glad to know when he may equitably receive the whole 522l. allowing 5 per cent. per annum simple interest for the first sum, and rebating at the same rate for the last.

Answer. The author's solution is, the payment must be April 30, 1721.

Answered by Mr. J. Jope, jun. taken from the Introduction to the Diary for 1722.

Put n for the time after Michaelmas 1719. Then $100 : 5 :: 320 : 16$, and $1 : 16 :: n : 16n$, the interest of 320l. Then $320 + 16n =$ the amount of 320l. $5 - n =$ the time the 202l. was received before due. $1 : 5 :: 5 - n : 25 - 5n$ and $125 - 5n : 100 :: 202$ $: \dfrac{20200}{125 - 5n} =$ value 202l. at the day of payment. Then by the question $16n + \dfrac{20200}{125 - 5n} = 202$, which by reduction gives $n =$ 1·760076 $= 1$ year 277 days 10 hours. So that the day of payment should be July 4th, 1721.

Additional Solution.

The sum 320l. being due the 29th of September, and the other sum 202l. due just five years afterwards; the meaning of the question is to find such a day between these two times, as that the

whole sum 522l. being then received, neither the receiver nor de-
liverer may suffer any loss ; in which case it is evident that the in-
terest of the 320l. for the time it is kept beyond the time when it was
due, must be equal to the discount of the 202l. for the time it is re-
ceived before it is due.

Hence then denoting the time for the discount by x, and conse-
quently the time for the interest, or till the whole must be paid, by

$5 - x$; by the rules for interest and discount, $320 \times \frac{5}{100} \times (5 - x)$
$= 16 \times (5 - x)$ will be the interest of 320 for the time $5 - x$, and
$100 + 5x : 5x :: 202 : \dfrac{202x}{20 + x} = $ the discount of 202 for the
time x, the rate of interest being 5 per cent. Consequently $16 \times$
$(5 - x) = \dfrac{202x}{20 + x}$; hence $x^2 + 27\frac{1}{8}x = 100$, and $x = 3.239888$.
Wherefore $5 - x = 1.76$ years nearly $= 1$ year 277 days ; which
being added to Sept. 29, we have the 3d of July, 1721, for the day
of payment. II.

II.   QUESTION 76, by Mr. W. Crabb.

If the side of the face of each of the regular solids (or platonic
bodies as they are called) be 29 inches : What is the content of each
in wine gallons.

Answered by Mr. Dodd.

The dodecaedron contains 809·072, the icosaedron 230·343, the
hexaedron 105·58, the octaedron 49·77, and the tetraedron 12·442
wine gallons.

Solution.

By page 404, &c. of my Mensuration, it is found that the solidities
of the five regular bodies, the side of each face being 1, will be thus :

$$
\begin{array}{lcl}
\text{Tetraedron} & = & 0.11785113 \\
\text{Hexaedron} & = & 1.00000000 \\
\text{Octaedron} & = & 0.47140452 \\
\text{Dodecaedron} & = & 7.66311896 \\
\text{Icosaedron} & = & 2.18169499.
\end{array}
$$

Then, since similar solids are as the cubes of their like sides, each
of these numbers being multiplied by 24389, the cube of 29, the pro-
ducts will be the contents of the bodies in inches ; and if these contents
be divided by 231, the inches in a wine gallon, there will result the
several numbers of gallons as in the original solution.   II.

G 2

III. QUESTION 77, by Mr. W. Deare.

In the triangular field ABC, let BP be perpendicular to the base; then there are given the vertical angle ABC $=120°$, the side AB $=8$ chains, the alternate segment of the base PC $=5\frac{1}{3}$ chains: Required the other side BC, the remaining segment AP and the area?

Answered by Mr. Crabb, Mr. Finch, Mr. Gurney, Mr. Dodd, and Mr. White.

24·2787 chains $=$ 2 acres 1 rood 28 perches.

Solution.

Put $a = $ BC $= 8$, $b =$ AP $= 5\frac{1}{3}$, $s = \frac{1}{2}\sqrt{3}$ sine of the angle ABC or $120°$, and $z =$ PC. Then, by right-angled triangles, BP $= \sqrt{(a^2-z^2)}$, and AB $= \sqrt{(b^2 + a^2 - z^2)}$; AB $\times$ BC $\times$ sine $\angle$ B $=$ twice the area $=$ BP $\times$ AC, that is, as $\sqrt{(b^2 + a^2 - z^2)} = (b + z). \sqrt{(a^2 - z^2)}$; and this, by squaring both sides, and writing the numbers instead of the letters, becomes $48 \times (16745 - 169z^2) = (77 + 13z)^2 \times (64 - z^2)$, which will be an equation of the 4th order; and by expanding it out, the root z may be found by converging series. But the root will be much easier found by the method of Trial and Error, from the equation as it here stands; and by this method z will come out 7·077 or $7\frac{1}{13}$. Hence the base AC is 13, and the area 24·248 square chains.
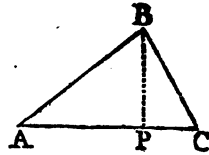
Scholium. This problem is the same as to divide a given angle into two parts such, that the tangent of the one may be to the secant of the other, in a given ratio. H.

IV. QUESTION 78, by Mr. G. Hare.

In latitude $53°$ north, a plane declines north-west $43° 15'$ and reclines backwards $49° 20'$; now suppose this plane to be carried parallel to itself along a meridian till it becomes upright; it is then required to describe a sun dial upon it?

Answered by Mr. Hawney, *Mr.* Failes, *Mr.* White, *Mr.* Williams, *and Mr.* Finch.

h.	equin.		h. dist.	
	°	′	°	′
VIII	140	9	143	12
IX	125	9	128	20
X	110	9	112	24
XI	93	9	84	13
XII	80	9	78	56
I	65	9	62	33
II	50	9	46	53
III	35	9	32	7
IIII	20	9	18	8
V	5	9	4	36
	Sub—		Stile E.	
VI	9	51	8	48
VII	24	51	22	26
VIII	39	51	36	38

	°	′
The dial's new latitude south..	4	58
New declination west	26	31
Dist. of substile and merid.....	78	56
The stile's height............	63	3
Distance of merid. and hor.....	54	29
Plane's differen. longit.	80	9

Solution.

Construction. Let ELQN be the horizon and z the zenith of the place in lat. 53°, P the pole, LZN the position of the plane making the angle EZL = 43° 15′, EZQ being the prime vertical, PZDA the meridian, and EDQ the equinoctial. Perpendicular to LN draw ZB equal to 49° 20′ the reclination, and draw the great circle LABN, which will be the plane of the dial, A being the new place required; also draw the great circle or meridian CPF perpendicular to NALF.
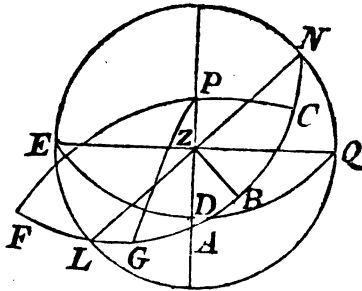
Then DA will be equal to the new latitude south; in which the plane will be upright, the angle ZAB the co-declination, AF the distance of the substile from the meridian, FC the stile's height, AL the distance of the meridian and horizon, and the angle APF the plane's difference of longitude.

Calculation.

1. In the right-angled triangle ABZ, given ZB the reclination = 49° 20′, and ∠AZB the declination = 43° 15′. Then,

As cos. ∠AZB = 43° 15′ : radius :: tang. ZB = 49° 20′ : tang. AZ = 57° 58′; from which taking ZD = 53°, there remains DA = 4° 58′ the new latitude.

G 3

As radius : sine ∠AZB :: cos. ZB : cos. ∠A = 63° 29', whose com-
plement 26° 31' is the new declination.
As radius : sine ZB : : tang. AZB : tang. AB ≍ 35° 31', the comple-
ment of 54° 29' the dist. of merid. and horiz.

2. In the right-angled triangle AFP, are given AP = 94° 58', and
·∠A = (180° — 63° 29' =) 116° 31'. Then
As radius : cos. ∠A :: tang. AP : tang. AF = 78° 59' the distance
of the substile from the merid.
As radius : sine AP :: sine ∠A : sine FP = 116° 57', whose supple-
ment 63° 3' is the stile's height.
As radius : cos. AP :: tang. ∠A : cotang. ∠P = 80° 9' the plane's
diff. of longitude.

Now this 80° 9', or plane's difference of longitude shews that the
distance, on the equinoctial DE of the hour line PA of 12, is 80° 9'
from the substile PF; and which is therefore set opposite to it in the
table to the original solution. The equinoctial distances in the same
column of the table belonging to the other hours, are found by the
continual addition and subtraction of 15°. And the several dis-
tances on the plane of the dial in the other column of the table,
against the equinoctial distances are all found by this proportion, As
radius : sine 63° 3' the stile's height :: tang. of each equinoctial dis-
tance : tang of the corresponding distance on the plane of the dial.
For if PG be any hour circle; then in the right-angled triangle PFG,
As radius : sine PF : : tang ∠P : tang. opposite side FG.

Note. In the table to the original solution, for the distance on the
plane against the hour XI the author has set 84° 18' instead of its
supplement; or rather indeed it should be 95° 46' the sup. of 84° 14'
as appears by calculation. H.

V. QUESTION 79, *by Mr.* John Ashton.

Upon the first of May a plane mirror being placed horizontally, at
the middle of the south side of a room, it reflected the sun's rays
exactly with the upper part of the east corner, just when the sun's
azimuth was equal to its altitude.— Required the dimensions of the
room, and the time of the day when the observation was made, the
area of the end of the room being 80 square feet?

◆   *Answered.*

This question was imperfectly proposed, no latitude being fixed,
so answers may vary for different places. The proposer in latitude
53° 28', as also Mr. *Crabb*, Mr. *Allen*, Mr. *Finch*, Mr. *Wall*, Mrs.
Dod, and several others, answer.

Time of the day 2ʰ 6ᵐ altitude and azimuth 48° 20', length of the
opposite side of the room 7·29216 feet, height 10·97, breadth of end
walls 3·245, May 1, 1719, London.

Solution.

With the above original answer, supposing the latitude to be 51° 32′ that of London. Then, in the annexed figure, if P be the pole, z the zenith, and ☉ the sun; we have given PZ $=$ 38° 28′ the co-latitude, P☉ $=$ 72° 6′ the co-declination for May 1, old stile, or May 12, new stile, and the supplement of the angle z $=$ the complement of z☉; to find the hour angle P.
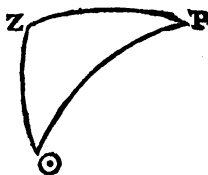
Put a and $b =$ the sine and cosine of PZ, $c =$ the cosine of P☉, and x and $\sqrt{(1 - xx)} =$ the sine and cosine of z☉, or cosine and sine of the $\angle z$.

Then, by common trigonometry, $axx + b \sqrt{(1 - xx)} = c$. Hence by completing the square and extracting the roots we obtain $x =$ ·6937742 $=$ the cosine of 46° 4′ the altitude and azimuth. Also, as sine P☉ : sine z :: sine z☉ : sine P $= (x \div d) \sqrt{(1 - xx)}$ putting $d =$ sine P☉ $=$ ·5250344 $=$ the sine of 31° 40′; which, at 15° to the hour, answers to $2^h 6\frac{2}{3}^m$ afternoon.

Again, if ABCD represent the floor of the room, s the speculum in the middle of the south side DC, and B the east corner, and SB be drawn; then will the angle B be $=$ 46° 4′ $=$ the azimuth from the south; also the triangle formed by SB, the height of the room, and the re-flected ray, will be similar to the triangle SBC, because the altitude is equal to the azimuth. Wherefore, if s and c denote the sine and cosine of 46° 4′ the azimuth, or the sines of the acute angles of the said two triangles, and z the line SB; then will SC be $= sz$, CB $= cz$, and $c : s :: z : sz \div c =$ the height of the room. Now if the area of the end BC be 80, then $(sz \div c)cz = sz^2 = 80 = a$; hence $z = \sqrt{(a \div s)} = 10\cdot54$, and then the breadth BC $= 7\cdot3123$, the length DC $= 15\cdot1804$, and the height $= 10\cdot94$.
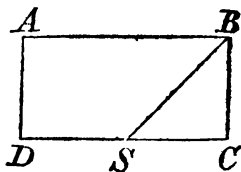
But if the area of the side DC be 80; then $(sz \div c)2sz = \dfrac{2s^2z^2}{c} = a$; hence $z = \sqrt{(\frac{1}{2}ac)} \div s$, and then the breadth $= 5\cdot075$, the length $= 10\cdot536$, and the height $= 7\cdot5931$.   H.

VI. QUESTION 80, by S. T.

Suppose a hollow sphere equal to the globe of the earth; and suppose a current of water 12 yards broad, and 6 yards deep, to run at the rate of a thousand yards an hour; in what time would the cavity be filled by the current?

Answered.

In regard the diameter of the earth is variously computed, the answers to this must differ from one another: accordingly Mr *Crabb* answers 1469,982,847,541 years according to 60 miles to a degree.

Solution.

By the question, $6 \times 12 \times 1000 = 72000$ cubic yards of water run in an hour, or $72000 \times 24 \times 365\frac{1}{4} = 631152000$ cubic yards in a year. Now if the circumference of the earth be 21600 nautical miles, then $\dfrac{21600^3}{(3\cdot14159 \&c.)^2 \times 6}$ nautical miles, or $\dfrac{21600^3 \times 1760^3 \times 7^3}{(3\cdot14159 \&c.)^2 \times 6^4}$ cubic yards will be the content of the earth; which being divided by 631152000, the quotient is 2334289000000 for the number of years required.

So that the first figure in the original answer seems to be falsely printed 1 for 2.   **H.**

VII.   QUESTION 81, *by Mr.* J. Jennings.

The expence of building a house which was afterwards sold, exceeded the builder's gain by 160l. The sum of the squares of its expence and the gain amounted to 70600l. Required the original cost and the gain ?

Answer.

Two hundred and fifty pounds cost, and ninety pounds gain.

First Solution.

In this question, are given the difference of two numbers, and the sum of their squares, to find the numbers.

Put $d = 160$ the difference, $s = 70600$ the sum of the squares, and $x =$ the cost. Then $x - d =$ the gain, and $x^2 + (x - d)^2 = 2x^2 - 2dx + d^2 = s$. Hence $x = \dfrac{d + \sqrt{(2s - d^2)}}{2} = 250$ the cost, and $x - d = 90$ the gain required.   **H.**

Second Solution.

Let x represent what the house cost, and y what was gained; also let $d = 160$ and $s = 70600$. Then, by the question,

$$x - y = d$$
$$x^2 + y^2 = s.$$

The square of the first equation taken from the second gives
$$2xy = s - d^2.$$
This added to the second, and the square roots taken gives
$$x + y = \sqrt{(2s - d^2)}.$$
But $x - y = d.$

Half the sum and half the difference of these equations give
$$x = \frac{\sqrt{(2s - d^2)} + d}{2}$$
$$y = \frac{\sqrt{(2s - d^2)} - d}{2}. \quad \text{L.}$$

VIII.   QUESTION 82, *by Mr.* William Doidge.

The slant side of a cone is 60 feet and solid content 30063 cubic feet: What is its perpendicular height?

Answer.

The perpendicular height 55·5234 feet.

Solution.

Put $a = 60$ the side, 30063 the content, and $z =$ the perpendicular height Then the radius of the base will be $\sqrt{(aa - zz)}$, the area of the base $= (aa - zz) \times 3\cdot1416$, and the content $= \frac{a^2z - z^3}{3} \times 3\cdot1416 = 30063$; hence $z = 55\cdot52446 =$ the perpendicular required.
H.

IX.   QUESTION 83, *by Mr.* J. Plomley, *or* N. M.

Suppose two travellers set out at the same time from Bristol to go round the world; and suppose one of them to go due east at the uniform rate of $7\frac{455}{21533}$ miles a day; and the other to go north at the rate of $11\frac{143}{4307}$ miles: how many times round the world will each have to go before they can meet again at Bristol: also required the number of days and the number of miles travelled by each?

Answer.

The first man travels 7 times round the globe, 151200 miles; the second, 11 times, or 237600 miles, and in 21535 days, or 59 years.

Solution.

It is evident that the number of times round each person travels, be-

fore they meet, will be directly as his rate of travelling. The ques-
tion then only requires to be found the least two whole numbers that
shall be in a given ratio; and such two numbers it is clear must be
the least integer terms of the said ratio. Now the given terms of the
ratio are $11\frac{143}{4307}$ and $7\frac{455}{21535}$ or $7\frac{91}{4307}$; and by dividing the former
of these terms by 11, and the latter by 7, the quotient of each is
$1\frac{13}{4307}$; therefore the least terms of the ratio are 7 and 11, and which
of consequence give the number of times each travels round.   H

<div align="center">THE PRIZE QUESTION.</div>

There is a kneading trough, the bottom of which is a rectangle;
its perimeter is 11·75 feet, and area 5·875 square feet: the sides,
ends, and bottom have all the same breadth; and the diagonal of a
section, made by a plane at right-angles to the longest edge of the
bottom is equal to $\frac{4}{5}$ of the breadth of the bottom; required the con-
tent of the whole trough in wine gallons, also the content of each of
the wedges made by a plane passing through the longest edge of the
bottom and the opposite edge, or brim of the top?

<div align="center">*Answered by Mr. J. Jope, jun.*</div>

Length at top	5·591	Diagonal	2·1300
Breadth	2·272	The whole vacuity	18563·74
Length at bottom	4·5969	Vacuity even with diag.	12156·85
Breadth	1·2780	Remaining part	6406·89
Depth	1·1774	Liquor,	27·735 wine gallons.

<div align="center">*Solution.*</div>

Let ABCDEHGF be the trough, its depth DI, being a perpendicular
from top to the bottom. Then by
comparing the words of the ques-
tion with this figure, the conditions
will appear to be thus : Of the rect-
angle EFGH the perimeter is 11·75,
and the area = 5·875; HE = EK = $\frac{4}{5}$ of
HK the nearest distance of GH from AD;
and HE, EK are also each = the dis-
tance between CD and HE, in con-
sequence of which EL = LI will be
equal to the half difference of both the
lengths and breadths at the top and
bottom, and therefore the solid is a
prismoid, and not the frustum of a
pyramid : And it is required to find
the whole content, as well as that of the wedge ADEHGF.
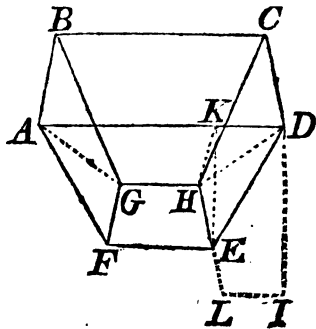
Now if x and y denote the length and breadth of the bottom, then $2x + 2y = 11\cdot75$, and $xy = 5\cdot875 = x + y$. From these equations we find $x = 4\cdot597 = $ EF, and $y = 1\cdot278 = $ EH $=$ EK. And therefore HK $= \frac{3}{2}$EH $= 2\cdot13$. Then in the obtuse isosceles triangle HEK, by Euc. II. 12, EL $=$ LI $= \dfrac{\text{KH}^2 - \text{KE}^2 - \text{EH}^2}{2\text{EH}} = \dfrac{\frac{2}{9}\text{HE}^2 - 2\text{HE}^2}{2\text{HE}} = \frac{7}{18}$HE $= \cdot497 = $ half the difference between the lengths and breadths; and whose double therefore being added to the length and breadth at bottom, we have $5\cdot591 = $ AD, and $2\cdot272 = $ DC, the length and breadth at top. Also, by right-angled triangles, DI $=$ KL $=$

$$\sqrt{(\text{KE}^2 - \text{EL}^2)} = \sqrt{\left(\text{HE}^2 - \frac{7^2\,\text{HE}^2}{18^2}\right)} = \frac{5\text{HE}\sqrt{11}}{18} = 1\cdot1774 = \text{the}$$

depth of the vessel.

Then, by page 161 Mensuration, the content of the wedge ADEHGF is $\dfrac{2\text{EF} + \text{AD}}{6} \times$ EH $\times$ DI $= 3\cdot707873$ cubic feet, or $= 6407\cdot204$ inches $= 27\cdot728$ wine gallons. And the content of the wedge ABCDHG $= \dfrac{2\text{AD} + \text{GH}}{6} \times$ DC $\times$ DI $= 7\cdot034942$ cubic feet $=$ $12156\cdot37$ inches. Also the sum of these two is $10\cdot742815$ feet $=$ $18563\cdot57$ inches. H.

——◦◦●◦◉◦●◦◦——

Questions proposed in 1721, *and answered in* 1722.

I. QUESTION 84, *by Mr.* Sam. Dicker.

In the city of Megara there were two small square temples dedicated to Bacchus and Venus, their floors were paved with stones each a foot square. The number of stones in both amounted to 2120; but each side of the floor of the temple of Venus exceeded that of the temple of Bacchus by 12 feet; what were their dimensions?

Answered by Mr. Cha. Glover.

Let $a = $ the greater side, $e = $ lesser. There's given the difference of the sides d, and the sum of their squares z; which by involution, subtraction, and evolution, this equation is found

$$a = \frac{d + \sqrt{(2z - dd)}}{2};$$ so $a = 38$, $e = 26$ the solution.

Additional Solution.

Put $z =$ the half sum of the two numbers, $d = 6$ the half difference, and $s = 2120$ the sum of their squares. Then $z + d$ and $z - d =$ the two numbers; the sum of whose squares is $2\,zz + 2\,dd = s$.

Hence $z = \sqrt{(\frac{1}{2}s - dd)} = \sqrt{(1060 - 36)} = 32$; and therefore 38 and 26 the two numbers. H.

II. QUESTION 85, *by Mr.* Moyle.

The angle of altitude of a cloud was observed to be 20°, and that of the sun in the same direction 35°; and the distance of the shadow of the cloud from the station of the observer measured 2304 yards; Hence the perpendicular height of the cloud above the ground is required?

Answer.

The cloud's height is 1747 yards by the prop. By some tables and rules it is 1744·8; but taking no notice of the ☉'s alt. at the different places, 'tis 1746·3.

Solution.

Let s be the sun, c the cloud, and o the observer, all in the same vertical plane; upon the horizon or let fall the perpendicular cp the height of the cloud; draw sc, and produce it to cut the horizon in A the shadow of the cloud. Then oA will be given $= 2304$ yards, the angle AOC $= 20°$; and if the elevation CAP of the sun at A be supposed $=$ the elevation sop at o, then the angle CAP $= 35°$, the $\angle$ OAC $= 145°$, and $\angle$ OCA $= 15°$.——Hence, as s. $\angle$ ACO : OA :: s. $\angle$ AOC : AC $=$ $\dfrac{\text{s. }20°}{\text{s. }15°} \times$ AO; and, in the triangle APC, as 1 (rad.) : s. $\angle$ CAP :: AC
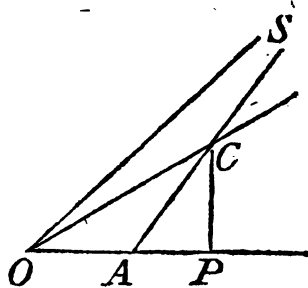
: CP $= \dfrac{\text{s. }20° \times \text{s. }35°}{\text{s. }15°} \times$ AO $= 1746\cdot34$ yards, or near a mile for the height of the cloud.

But as 2304 are rather above 1 minute of a degree of a great circle, the $\angle$ CAP will be about 35° 1′, and then CP $= \dfrac{\text{s. }20° \times \text{s. }35° \, 1′}{\text{s. }15° \, 1′}$

$\times$ AO $= 1745\cdot17$ yards the distance, a little less than before. H.

III. QUESTION 86, by Mr. Jos. Dogharty.

Three ships sailed from the same place to different ports in the same parallel of latitude: the first sailed directly south 55 leagues, when she arrived at the desired port; the other two sailed upon different courses, between the south and west, till they arrived at their destined ports, which were 57 leagues asunder, and the angle included by their courses at the port sailed from was 38°; required the course and distance run by each of the two latter vessels. Or in other words, given the base, the perpendicular, and the vertical angle of a triangle to find the rest?

Answered by Mr. John Jope, *jun. of Loo, in Cornwall.*

$x^4 + 2ax^3 + a^2 + 2b^2x^2 + 2ab^2x = a^2b^2R^2 \div c^2 - a^2b^2 - b^4$. Where x is found $= 15{\cdot}103$: whence the course is S. 15° 21′ W. and the distance $= 57035$ leagues. The other course is S. 53° 31′ W. and the distance 92·139 leagues.

The above expression is wrong printed: it should be $x^4 + 2ax^3 + (a^2 + 2b^2)x^2 + 2ab^2x = a^2b^2R^2 \div c^2 - a^2b^2 - b^4$; as appears from the following solution.

Let the angle OCA (see the last figure) $= 38°$, be denoted by φ, OA $= 57 = a$; CP the perpendicular, $= 55 = b$, and AP $= x$.

Then OC $= \sqrt{(x^2 + 2ax + a^2 + b^2)}$ and DC $= \sqrt{(x^2 + b^2)}$.

Now the radius being supposed unity, it is well known that the rectangle of the sides OC, CA multiplied by the sine of their included angle, gives twice ($\frac{1}{2}$OA × PC $= \frac{1}{2}ab$) the area of the triangle OCA:

Hence $\sqrt{(x^2 + 2ax + a^2 + b^2)} \times \sqrt{(x^2 + b^2)} \times \sin \varphi = ab$; or $x^4 + 2ax^3 + (a^2 + 2b^2)x^2 + 2ab^2x + a^2b^2 + b^4 = a^2b^2 \div \sin^2\varphi$; or $x^4 + 2ax^3 + (a^2 + 2b^2)x^2 + 2ab^2x + b^4 = a^2b^2 \div \sin^2\varphi - a^2b^2$; which is Mr. Jope's expression, (R being supposed $= 1$).

Taking square roots we get $x^2 + ax + b^2 = ab$ cotan φ: therefore, completing the square, and again taking the square roots we have $x =$

$$\sqrt{\left(\frac{a^2}{4} + ab \text{ cotan } \varphi - b^2\right)} - \frac{a}{2} = 13{\cdot}92484.$$ Hence, PC : rad. ::

AP : tan PCA, therefore the angle PCA is 14° 12′ 27″; and, as sin PCA : PA :: rad. AC $= 56{\cdot}735$ miles: Also, as sin AOC : PC :: rad. : OC $= 89{\cdot}751$. L.

Another Solution.

Let c, P, A, o (see the last fig.) be the four ports; then CP $= 55$ leagues, AO $= 57$ leagues, and $\angle$ ACO $= 38°$. Now, by prop. 13,

Simpson's Trigon. As 1 to cot. $\frac{1}{2}$ $\angle$ ACO :: 2AO $\times$ CP : (OC+CA)^2—AO^2; hence OC + CA = $\sqrt{}$(AO^2 + 2AO $\times$ CP $\times$ cot. $\frac{1}{2}$ $\angle$ ACO) = $\sqrt{}$(57^2 + 57 $\times$ 110 $\times$ cot. 19°) = 146·487 the sum of the sides. And, by prop. 14 of the same, 1 : tan. $\frac{1}{2}$ $\angle$ ACO :: 2AO $\times$ CP : AO^2 — (OC—CA)^2 ; and hence OC—CA=$\sqrt{}$(AO^2—2AO $\times$ CP $\times$ tan. $\frac{1}{2}$ $\angle$ ACO)= $\sqrt{}$(57^2 — 57 $\times$ 110 $\times$ tan. 19°) = 33·016 the difference of the sides. Then, the half sum increased and diminished by the half difference, we have 89·7515 and 56·7355 for the two distances CO and CA. Again, as 146·487 (sum of sides) : 33·016 (diff. of sides) :: cot. 19° ($\frac{1}{2}$ $\angle$ OCA) : tan. 33° 12′ = half the difference between the angles A and O ; and from the half sum and difference those angles are found = 104° 12′ and 37° 48′ the complements of 14° 12′ and 52° 12′ the two courses required.

It is evident that this problem will be constructed, by describing on the given base AO a segment of a circle to contain the given vertical angle ; and then a line drawn parallel to the base at the distance of the perpendicular from it, will cut the circle in the vertex of a triangle.   H.

IV.   QUESTION 87, by Mr. T. Williams of Middleton-Stoney.

A person bought a quantity of land the bounds of which are to be marked out by the shadow of a tree 103 feet high, between the hours of eight and one on the 10th of march 1721 : now supposing the sun should not appear on that day, how may the bounds and area be determined the latitude being 52° N.

Answered.

	hours	⊙'s alt.		azim.		leng. shade
The 10th of March.	VIII	17°	56′	65°	32′	318·2
Latitude 52.	IX	25	48	51	45	212·9
	X	32	13	36	14	163·4
	XI	36	29	18	47	139·23
	XII	38	0	0	0	131·83
	I	36	29	18	47	139·23

Feet.

. The figure will be triangular, the shadow at 8 and 1 being two of its sides, and right lines; the third (the path of the nodus or top of the tree) a curve line, of the conic sections, about 334 feet. The content 22200 square feet = 210. 1 per. and 148 square feet. Those who have taken the horizontal distances with respect to the pole, instead of the azimuths, have given false solutions.

This Question may be wrought thus:

As rad............	90°	0′	As cotang. lat. 52° 0′			9·89280
To cosine lat.	52′	0	To radius			10·00000
So is cosine of the hour	60	0	So tang. altit. ...17 56			9·51005
To sine of the altitude	17	56	To sine azimuth 65 32			9·61724

As sine 17° 56′ : log. 103 :: sine 72° 4′ : log. 318, and so for the rest. And you'll have several triangles with 2 sides and an angle between them given, to find their contents; which in one sum gives the area of the whole more exact than by taking it as one single triangle, by reason of the curve line in each of them.

Additional Solution.

The piece of land in this question is a triangle on the horizontal plane, of which two sides are the shadows of the tree at 8 and 1 o'clock, and the third side the line described by the shadow of the top of the tree between the same two hours; and this third line will be right as well as the two former, and not a conic section, because the sun is in the equinoctial, it being the 10th of March old stile. The said third line will also be directly east-and-west, or perpendicular to the meridian or shadow of the tree at 12 o'clock.

Now, if P represent the pole, Z the zenith, and ☉ the sun at any hour: Then are given PZ the co-latitude, P☉ a quadrant, and ∠P = the hour from 12; to find the ∠PZ☉, the sun's azimuth or bearing at that hour. And, taking the extreme hours 8 and 1 in the question, their azimuths will come out 65° 32′, and 18° 47′; which are the two angles ABP, CBP contained by the meridian shadow BP, and the two extreme shadows BA, BC, the place of the tree being at B. Now the sum of these two angles is 84° 19′ = the ∠ABC, and their complements are 24° 28′ and 71° 13′ = the angles A and C. So that all the angles are then known.
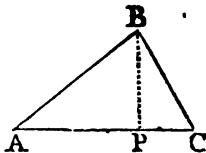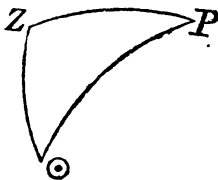
Again, the meridian altitude of the sun being 38° = the complement of the latitude 52°, we shall have as 1 : tang. 52° :: 103 feet (the height of the tree) : 131·834 = BP the length of the meridian shadow. From hence, and the given angles, AP and PC are easily found = 289·73 and 44·837, and then the area = 22053·65 square feet = 2·0218 roods. H.

v. QUESTION 88, *by Mr.* Chris. Harris.

A piece of land in the form of a regular heptagon was purchased

for as many shilling as would surround it touching one another, the diameter of each being one inch : the number of shillings which one acre cost was equal to eleven times the number of acres inclosed : How many acres were purchased and what price per acre?

Answered by Mr. L. Evan.

If 3·633959 be the area of a heptagon, whose side is 1; then put $x =$ the acres, $11x =$ shillings, the price of 1 acre. $1 : 11x ::$ $x : 11xx =$ the price in shillings of all the acres bought $=$ inches round the polygon ; 6272640 inches $=$ 1 acre.

$3·633959 : 1 :: 6272640x : \dfrac{6272640x}{3·633959} =$ the square of each side.

7 times the square root of that $= 11xx$. That is $\sqrt{\dfrac{6272640x}{3·6339}} =$

$\dfrac{11xx}{7}$, or $\dfrac{6272640x}{3·6339} = \dfrac{121x^4}{49}$, or $\dfrac{307359360x}{3·633959} = 121x^4$; hence

$307359360x = 439·709039x^4$, and $699006·235 = xxx$. Then $88·74999 = 88$ ac. 2 ro. 39 per. $= x$ the number of acres bought. $976·24989$ shill. $= 48l.$ 16s. $=$ the price of 1 acre.

Additional Solution.

Since the price of one acre in shillings is equal to 11 times the number of acres, it is plain that the price of all the acres will be 11 times the square of the number of acres ; but the price of the whole is equal to the circumference in inches, therefore the circumference in inches is equal to 11 times the square of the acres.

Now if x denote the side of the heptagon in inches, $a = 3·633912$ the area of a heptagon whose side is 1, and $b = 6272640$ the inches in an acre. Then $7x$ will be the perimeter, and $ax^2 \div b$ the acres; wherefore $7x = 11a^2x^4 \div b^2$; hence $x = \sqrt[3]{(7b^2 \div 11a^2)} = 12377·1$ $=$ the side. Then $ax^2 \div b = 88·7487$ the number of acres. H.

VI.   QUESTION 89, by Adrastea.

Twenty four persons, consisting of captains, mates, sailors and boys, agreed to pay 21d. for a passage by sea, in the following proportions : a captain was to pay a certain sum, a mate half as much as a captain, a sailor one fourth as much as a mate, and a boy half as much as a sailor: how many of each were there, and what did each pay for his passage?

Answered.

This question is unlimited.

For when the number of quantities sought exceeds the number of given equations, the question is capable of innumerable answers.

$$24 \begin{cases} \text{2 captains 4l. each } = 8l. \\ \text{4 mates \quad 2} \ldots \ldots = 8 \\ \text{14 sailors \quad 10s.} \ldots \ldots = 7 \\ \text{4 boys \quad 5} \ldots \ldots : = 1 \end{cases} \begin{cases} l. \\ 24 \end{cases}$$

Mr. *Evans* collected 100 true answers, which for brevity I omit.

Remark. In the above original answer, it is justly remarked that the data are insufficient; for, to have limited the answers, there ought to have been three more conditions; so if three of the numbers of persons had been given, the question would have been confined to one answer, and the method of solution too easy to need any pointing out here. H.

VII. QUESTION 90, *by Mr.* Deare.

A gentleman dying left an estate of 50000l. to his two sons : the square of the pounds in the elder son's share was to be equal 50000 times the youngest son's share : What was the share of each ?

Answered by Mr. Cha. Glover.

Let $a =$ eldest son's portion, $e =$ youngest, $aa = se$, and $s - a = e$ per quest. this equation is produced,
$a = \sqrt{(ss + \frac{1}{4}ss)} - \frac{1}{2}s$. $a = 30901{\cdot}69943746$ or $30901l.$ $14s.$
$e = 19098{\cdot}30056251$ or $19098l.$ $6s.$

Additional Solution.

In this question are to be found two numbers, of which the sum is given, and the square of the greater equal to the less multiplied by the given sum. Or, it is required to divide a given number into two such parts, that those parts and the whole may be three numbers in geometrical progression. Or, in other words again, to divide a given number according to extreme and mean proportion; which it is well known will not admit of an answer in integer numbers.

Put $s = 50000$ the sum, and x the greater part. Then $s - x =$ the less, and $xx = ss - sx$; hence $x = \dfrac{\sqrt{5} - 1}{2} \times s = 30901{\cdot}6994375$, and consequently the other part $= 19098{\cdot}3005625$, as above determined. H.

VIII. QUESTION 91, *by Mr.* C. Mason.

How must a horse be tethered so as to be able to graze just an acre

in the form of an ellipse, the ratio of the principal axes being as 7 to 9?

Answered by Mr. J. Andrew.

The transverse diameter 16·184053 perches = 89·01 yards.

The conjugate diameter 12·587597 perches = 69·23 yards. Distance between the focus points 55·9463; thence to the extreme parts 44·505, doubled 89·01 added to the distance between the foci is the length of the tether 145 yards.

The way for the horse to graze just an elliptical acre, is thus: Set up two stakes or pins in the longest diameter of the oval, 16 yards 6 inches and a quarter from the outside. Put a string of 145 yards long round both stakes, and tie the two ends together; at which knot let the horse's mouth be fix'd: then in going about he will exactly sweep the oval, containing 4840 square yards, or one acre.

Solution.

Put $9x$ and $7x$ for the two axes in yards. Then $63xx \times ·785398$ &c. $= 4840$, the square yards in an acre. Hence $x = \sqrt{\dfrac{4840}{63 \times ·7854}} =$ 9·890255; consequently $9x$ and $7x$ become 89·0123 and 69·2318 for the two axes. Then $89·0123 + \sqrt{(89·0123^2 - 69·2318^2)} = 144·96$ = length of the tether. H.

THE PRIZE QUESTION. *The gardens of Alcinous, from the seventh book of Homer's Odysseys, translated by Mr. Pope.*

Close to the gates, a spacious garden lies,
From storms defended, and inclement skies.
Four acres was th' allotted space of ground;
Fenc'd with a green inclosure all around.
Tall thriving trees confess'd the fruitful mould,
And red'ning apples ripen here to gold.
Here the blue fig with luscious juice o'erflows;
With deeper red the full pomegranate glows.
The branch here bends beneath the weighty pear;
And verdant olives flourish round the year.
The balmy spirit of the western gale
Eternal, breathes on fruits untaught to fail.
Each dropping pear, a following pear supplies;
On apples, apples; figs on figs arise:
The same mild season gives the blooms to blow,
The buds to harden, and the fruits to grow.

Here order'd vines, in equal ranks appear,
With all th' united labours of the year :
Some to unload the fertile branches run,
Some dry the black'ning clusters in the sun;
Others to tread the liquid harvest join.
The groaning presses foam with floods of wine.
Here are the vines in early flow'r descry'd,
Here grapes discolour'd, on the sunny side,
And there in autumn's richest purple dy'd.
   Beds of all various herbs, for ever green,
In beauteous order terminate the scene.
Two plenteous fountains the whole prospect crown'd,
This thro' the gardens leads its streams around,
Visits the plants, and waters all the ground;
While that in pipes, beneath the palace flows,
And thence its current to the town bestows :
To various use, their various streams they bring,
The people one, and one supplies the king.

The king's pipe delivers the water ten foot and an half (English measure) below the surface of the water in the fountain, by an inch and three quarters bore: His brewing once a week, takes fifty hogsheads of water, (wine measure) kitchen, landry, other offices, &c. five hogsheads a-day. The people have occasion for twenty hogsheads a-day; their cock of two inches diameter, is below the fountain twenty-two foot.   His majesty so much delights in a morning walk to see the waters undisturbed, that his orders are, that the cocks shall but run once every day and that in the evening; so long only, as may supply next day's consumption, and for so much in his reservoir, as supplies his brewing each week.   Then if the velocity or motion of the water be equal to that of an heavy body acquired in these descents; and also, it be premised, that heavy bodies accelerate as the squares of their times, as sixteen foot the first second it falls, &c.   How long every day ought each pipe to run, to give the king and people their due quantity?

Answer.

In answering this question in hydrostatics, the philosophy of the gravity and pressure of fluids is to be considered: That at the first moment the cock or adjutage is opened, the liquid flows out with the same velocity as a heavy body moves when fell from that height the liquor came, or the place of the reservoir.   The water that enters the top, or mouth of the pipe, moves as fast as that flowing out, at

what depth soever, in an equal cylinder. Whereas a heavy body
moves slow at first setting out, and continually receives a new im-
pulse of gravity, which when it has fallen the length of the water-
pipe, is equal in velocity to that column of water. Then as a body
accelerates with the odd numbers 1, 3, 5, &c. the water going out as
fast the first space, as a weight falls in the second; consequently must
be as Gravesand, in his Mathematical Philosophy, page 188, No 378,
says,

" In the time in which a body falling freely, goes through the
" height of the liquid above the hole, a column of the liquid flows
" out equal in length to twice that height."

On this principle was the question composed, having Gravesand then
before me; which is plain from the question itself, the word [acquired]
meaning no other, than that the water moved as fast as the body, when
it had fallen the height of that liquid: But the design of putting it in
those words was to prevent such who had not skill enough in philoso-
phy, stamping a solution to it.

The king used 765 gall. = 12¼ hogsheads per day. The bore 1·75
inch squared × by 252 double height ÷ 294·12 gives 2·6579 w. g.)
765 (287·9 times (the height) for the quantity. □ 60‴ = 3600)
192 (·05333 inch, the space accelerated in 1 third. Then 10½ feet
= 126 inches — ·05333 gives 48·621 thirds multiplied by 287·9 is
13994·819 thirds = 3′ 53″ 14‴. *The answer.*

The people use 1260 gallons per day.

Height dupla 528 × 4 dia.sq. ÷294·12 = 7·18 w. g.

·1260 ÷ 7·18 = 175·49 such columns √(22 ÷ 16) = 1·17 ×
175·49 = 205·3233 = 3′ 25″ 32‴. *The answer.*

Additional Solution.

Put a = the altitude of the surface of the water above the deli-
vering pipe, n = the area of the orifice or pipe, and $m = 32\frac{1}{6}$ feet
= 386 inches. Then proceeding as in page 4 of our new Math.
Miscel. $\sqrt{2ma}$ will be the velocity of the issuing water, upon the
supposition of the question, that it is equal to that of a heavy body
after falling though the space a; but if, according to Sir I. Newton
and some others, the velocity be that which is acquired by falling
through only $\frac{1}{2}a$, it will be $\sqrt{ma}$; which is to the former as 1 to
$\sqrt{2}$, and therefore the time for the one supposition may be found
from that of the other, by this proportion of 1 to $\sqrt{2}$.

Now in the former case with the velocity $\sqrt{2ma}$ per second, we
shall have $n\sqrt{2ma}$ for the quantity run out per second; and there-

fore $n \sqrt{2ma} : \mathbf{Q} :: 1$ second $: \dfrac{\mathbf{Q}}{n\sqrt{2ma}}$ = the seconds in which the

the quantity $\mathbf{Q}$ will be voided.

Then, for the king, $\mathbf{Q}$ = 12¼ hogsheads = 12¼ × 63 × 231 or
176715 cubic inches, $n = 1·75^2 × ·7854$, and $a = 10\frac{1}{2}$ feet = 126

inches; with which numbers $\dfrac{Q}{n\sqrt{2ma}}$ becomes 235·565 seconds, or 3^m $55\frac{1}{2}^s$.

And, for the people, $Q = 20$ hogsheads $= 291060$ inches, $n = 2^s$ $\times$ ·7854 $= 3\cdot1416$, and $a = 22$ feet $= 264$ inches; which numbers give 205·23 seconds $= 3^m$ $25\frac{1}{4}^s$; both nearly as in the original answer.

But if the other supposition be used, of $\sqrt{ma}$ velocity per second; then each of the above times must be multiplied by $\sqrt{2}$.   H.

Questions proposed in 1722, and answered in 1723.

I.   QUESTION 92, by Mr. J. Andrew.

Find the content, in ale gallons, of the greatest frustum of a cone which can be inscribed in a semi-spheroid, generated by the revolution of an ellipse whose principal axes are 30 and 20 inches?

Answered by Mr. C. Mason.

Let the semi-transverse 15 $= h$, and semi-conjugate 10 $= c$, length of the frustum $= a$, the difference put $= h - a$, and the semi-head diameter $= e$.   Then,

$$hh : cc :: hh - aa : \frac{hhcc - ccaa}{hh} = ee.$$ Which reduced gives

$a = 10\cdot606601$, and $e = 7\cdot075$, and $2e = $ less diameter $= 14\cdot15$ inches; hence the content in ale gallons $8\cdot6915$.

Mr. *Ri. Tapper*, by the doctrine of fluxions, gives this equation, $x^5 + \frac{5}{3}x^4 + \frac{4}{3}x^3 - \frac{2}{3}x = \frac{1}{3}$.   Here the flowing quantity $x = \cdot64638$. Hence this general theorem,

As 1 : the conjugate diameter of a spheroid :: ·64638 : the lesser diameter of the largest conic frustum that can be inscribed.   Whence

The lesser diameter............... 12·9277 } inches.
Height of the conic frustum.......... 11·4455 }
The solid content.................   8·7734 ale gallons.

But Mr. *Andrew*, the proposer, gives this general theorem, $x^6 - 425x^4 + 67500x^2 = 3796875$: and the lesser diameter $= 12\cdot9297$, and the content $= 15\cdot4587$ ale gallons.

H 3

Additional Solution.

To find the greatest frustum AEGC of a cone that can be inscribed in a given semi-spheroid ABC, the base of the cone being the same as that of the spheroid, whether it be oblate or oblong, that is whether the revolving axe AC be the less or the greater axe of the ellipse.
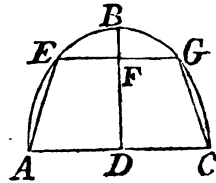
Put $r =$ AD the semi-revolving axe, $f =$ DB the semi-fixed axe, and $x =$ DF the altitude of the frustum. Then, by the ellipse,

$$ff : rr :: ff - xx : \text{EF}^2 = \frac{ff - xx}{ff} \times rr;$$

hence the solidity of the frustum will be

$$\left(rr + rr\sqrt{\frac{ff - xx}{ff}} + rr \times \frac{ff - xx}{ff}\right) \times \frac{3\cdot1416x}{3}, \text{ and therefore}$$

$$x \times \left(1 + \sqrt{\frac{ff - xx}{ff}} + \frac{ff - xx}{ff}\right) = \text{a maximum.}$$ —— Or, put

$zz = \dfrac{ff - xx}{ff}$, then $xx = ff \times (1 - zz)$, and the maximum $=$

$\sqrt{(1 - zz)} \times (1 + z + zz)$; which put into fluxions and reduced, we have the equation $1 + z - 2z^2 - 3z^3 = 0$; whose root z is $= \cdot6464881$ very exact. Hence $x = \sqrt{(1 - zz)} \times f = \cdot762924f$, and the content of the greatest frustum $= \cdot762924frr \times \dfrac{3\cdot1416}{3} \times$

$(1 + z + zz) = 1\cdot649344frr$.

Now in the case of the oblong spheroid, $f = 15$, $r = 10$, x the height $= \cdot762924f = 11\cdot44386$, and the content $1\cdot649344frr = 2474\cdot016$ inches $= 8\cdot773$ ale gallons.

But if the spheroid were oblate, then $f = 10$, $r = 15$, $x = 7\cdot62924$, and the content $= 3711\cdot024$ inches $= 13\cdot1597$ gallons.

Corollary. From this solution it appears, that the value of x, or the height of the frustum, will be always the same, whatever the revolving axe AC may be; the said height being to the semi-fixed axe, or DF to DB, as $\cdot762924$ to 1.——And the content $1\cdot649344frr$ of the frustum AEGC is to the semi-spheroid ABC or $2frr \times \dfrac{3\cdot1416}{3}$, as $2\cdot474016$ to $3\cdot14159$ &c. or as $\cdot787504$ to 1.——Also, the frustum in the oblate is to the frustum in the oblong semi-spheroid, as the longer axe to the shorter, which is the same proportions as the spheroids themselves are in by Cor. 8, page 277, of Mensuration.   H.,

II. QUESTION 93, *by Mr.* C. Mason.

Upon the top of a tower whose height is 253 feet, is a spire with a

vane at the top : On the 9th of June 1721, the sun's azimuth being
N.N.W. a person measured the length of the shadow of the tower
and vane upon the level plane, and found it exactly equal to the
height of the tower : Required the height of the vane above the tower,
the latitude being 54° ?

Answered by Sylvia.

If the bearing of the shadow be N.N.W. the sun must be upon the
S.S.E. point, his azimuth then being 22° 30′ from the south, and his
declination being the 9th June 23° 30′. In the latitude of 54° his
altitude is 58° 2′.

As the tangent of 45° 0′ .... 10·0000000
To the tangent of 58   2 .... 10·2047732
So is the logar.   253 foot.................... 2·4031204
To the logar.   403·4...................... 2·6078937

The height of the tower and vane together, but the vane's above the
tower just 152·4 foot.

Additional Solution.

In the spherical triangle zP⊙, given
PZ the co-latitude = 36°, P⊙ the co-
declination = 66° 30′, and the angle
z = the azimuth from the north =
157° 30′; to find the side z⊙ = 31° 58′
= the co-altitude. Hence the altitude
itself is 58° 2′ = ∠BAC in the second
figure, in which the shadow AB = the
tower BD = 253: then, by plane trigo-
nometry, as radius : tang. ∠BAC :: AB
: BC = the height of the tower and v ne
together ; from which taking away the
height of the tower BD, there remains DC
= the height of the vane as in the original
solution. H.
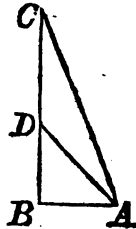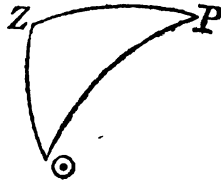
III.   QUESTION 94, by Mr. Richard Whitehead.

There is a triangular garden, in which is inscribed a circular hedge
of yew, which touches the three sides ; in the south side is a door,
just at the point of contact with the inscribed circle dividing the
side into two segments of 39½ and 30½ yards ; and if a perpendicular

H 4

is drawn to this side from the opposite angle, the rectangle of the segments made thereby will be 1181$\frac{3071}{9600}$ square yards; required the other two sides of the garden, together with its area; and also the diameter of the circle?

Answered by Mr. Ri. Tapper.

Let a = the greater segment taken from the perpendicular, e = lesser segment, and s = their sum = 70, p their product 1181$\frac{3071}{9600}$ yards. There comes out this equation,

$$a = \tfrac{1}{2}s + \sqrt{(\tfrac{1}{4}ss - p)} = 41\cdot6090925304966547$$
$$e = \tfrac{1}{2}s - \sqrt{(\tfrac{1}{4}ss - p)} = 28\cdot3909074695033453 \Bigg\} \text{ yards.}$$

Put x for the unknown tangent, h = the greater segment, d = the lesser, from whence this equation is brought out

$$x = \frac{dd + au - hh - ee}{18} = 16\cdot4040530149739870499445.$$

Hence the length of the sides are,

Longest 70
Next .: 55·904530149739870499445,
Shortest 46·9045 &c.

For the perpendicular's length, let g = the shortest side; then, by Eucl. I. 47, $\sqrt{(gg - ee)} = 37\cdot40536$ yards.

Hence the area = 1309·1876 square yards = 1 ro. 3 per. &c.

To find the angle subtended by the perpendicular, and made by the shortest side and shortest segment.

As the less segment : radius :: perpendicular : tang. 52° 48′.

As radius : tang. of $\frac{1}{2}$ last angle :: lesser segment 30·5 : 15·14, half the diameter; so the diameter is 30·28.

Remark. In this solution x is put = BH (in the figure to the following solution), b = AH or AE, and d = GB = CE, then AB = b + x and BC = d + x: but since AB² — BC² = AD² — DC², we have

$$b^2 - d^2 + 2(b - d)x = a^2 - e^2,$$

and therefore $x = \dfrac{d^2 + a^2 - b^2 - e^2}{2(b - d)}$ as in the original. L.

Additional Solutions.

In this question we have given, in the triangle ABC, the base AC = 70, the rectangle of the segments of the base, made by a perpendicular, AD × DC 1181$\frac{3071}{9600}$, and the segments of the base made at the point of contact of the inscribed circle, viz. AE = 39$\frac{1}{2}$, and CE = 30$\frac{1}{2}$; to determine the triangle.
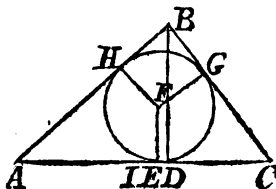
Algebraical Solution.

Put $s = 35 = $ AI $=$ IC, $p = 1181\frac{4071}{9800} = $ AD $\times$ DC, and $z =$ ID $=$ half the difference between AD and DC. Then AD $= s + z$, and DC $= s - z$; hence their product $ss - zz = p$, and therefore $z = \sqrt{(ss - p)}$. Then AD $=$ AH $= s + \sqrt{(ss - p)}$, and DC $=$ CG $= s - \sqrt{(ss - p)}$.——Again, since BH is $=$ BG, we shall have the difference of the sides AB $-$ BC $=$ AH $-$ CG $=$ AE $-$ EC $= 9$; then, by a known rule, AE $-$ EC : AD $-$ DC :: AC : AB $+$ BC; the sum of the sides being then known, from it and their difference the sides themselves are easily found. Also BD $= \sqrt{(\text{BA}^2 - \text{AD}^2)}$, and $\frac{1}{2}$BD $\times$ AC $=$ the area.——Lastly, the double area of the triangle being divided by its perimeter, the quotient will be the radius FE of the inscribed circle.

Geometrical Construction.

Let I be the middle of the given base AC, and E the point of con-tact with the circle. By Eucl. II. 14, make a square equal to the given rectangle of the segments of the base, which call s^2; then take ID equal one leg of a right-angled triangle of which s is the other leg and AI the hypothenuse; so shall AD, DC be the segments of the base made by the perpendicular.——For, by Eucl. II. 5, AI$^2 = $ AD $\times$ DC $+$ ID^2 $= s^2 + $ ID^2 by the construction; therefore AI, s, and ID form a right-angled triangle by Eucl. I. 48.

Again, take BG or BH a fourth proportional to 2IE, ED, and AC; which being added to AH $=$ AE, and to CG $=$ CE, the two sums will be the two sides AB, BC, of the triangle.——For, by Eucl. III. 37, AE $=$ AH, CE $=$ CG, and BH $=$ BG; and it remains only to prove that BG or BH is the 4th proportional mentioned above. Now, by Simpson's Geom. Cor. 2. 9. II, as AB $-$ BC : 2 ID :: AC : AB $+$ BC; but AB $-$ BC ($=$ AH $-$ CG $=$ AE $-$ EC) $=$ 2IE, and AB $+$ BC $=$ (AH $+$ HB $+$ BG $+$ GC $=$) AC $+$ 2BG; therefore the proportion becomes 2IE : 2ID :: AC : AC $+$ 2BG; hence, by division, 2IE : 2ED :: AC : 2BG, or 2IE : ED :: AC : BG.   *Q. E. D.*

The calculation from hence is very easy and evident.   H.

IV.   QUESTION 95, *by Mr.* Alex. Naughley.

To try the skill of a gauger he is required to find the content of a tub in the form of a frustum of a cone having given the bottom di-ameter 32 inches, and the segments of the diagonals made by their intersection 20 and 30 inches. He however finds that he is unable to resolve the question, and therefore solicits the aid of the learned fair to help him out of his difficulty.

Answered by Mrs. Eliz. Dodd.

The greatest diameter 48. Lesser 32. Depth 30 inches. Content 17·764834 malt bushels.

Solution.

ABCD being the tub, we have given DC $=$ 32, AE $=$ EB $=$ 30, and DE $=$ EC $=$ 20.

Upon the middle of the base of the isosceles triangle DEC let fall the perpendicular from the angle E; and from the top of the tub let fall the perpendicular BF. Then, in the right-angled triangle DEG, EG $= \sqrt{(DE^2 - DG^2)} = \sqrt{(20^2 - 16^2)} = 12$; and in the similar triangles DEG, DBF, as DE $= 20$: EG $= 12$ :: DB $= 50$: BF $= 30$. Also, by similar triangles, ED : DC :: EA : AB $=$ 48.
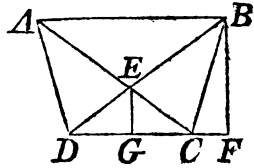
Hence the content $= ·7854 \times$ BF $\times \dfrac{AB^2 + AB \times DC + DC^2}{3} = 30 \times$

$\dfrac{48^2 + 48, \times 32 + 32^2}{3} \times ·7854 = 12160 \times 3·14159$ &c. $= 38201·77$

cubic inches $= 17·7648$ malt bushels. H.

V. QUESTION 96, *by Mr.* Joseph Smith.

There is a piece of land in the form of a triangle, the sum of the three sides is 96 chains; $\frac{1}{3}$ of the base or longest side is just equal to the difference of the other two ; and the difference of the segments of the base made by a perpendicular from the opposite angle is 2 chains less than the shortest side. The land is to be divided amongst three men, by fences made from the angles to the centre of gravity thereof: required the lengths of the fences ?

Answered.

The three sides of the triangle are 42, 34, and 20 chains. The internal fences 24·586, 18·773, and 12·238. The content 33·6 acres.

Divide each side of the triangle into two equal parts, from whence draw lines to the angles opposite; the point in which these intersect is the centre of gravity of the triangle; and where it would be, if hung up, equally poised.

Solution.

Let AC be the base or longest side of the proposed triangle ABC, and BD the perpendicular.——Put $z =$ BC the shortest side, and $y =$ the other side AB. Then, by the question, $z - 2 =$ AD — DC, and the greatest side AC (or AD + DC) $= 3.(y - z)$, also the sum of all the three sides is $4y - 2z = 96$, or $2y - z = 48$. But, by the nature of the triangle, AC : AB + BC :: AB — BC : AD — DC, or $3 . (y - z)$ $\times (z - 2) = (y + z) \times (y - z)$; hence $3 . (z - 2) = y + z$, or $2z - y = 6$. From this equation, and the one above, viz. $2y - z = 48$, we easily find $z = 20$, and $y = 34$. Consequently the third side $3 . (y - z)$ is 42.
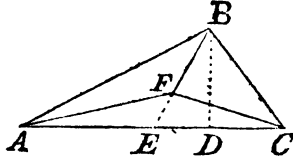
Now the point F upon which the triangle will be in equilibrium is the centre of gravity; and lines drawn through the centre of gravity, from the angular points, bisect the opposite sides; that is, BF produced bisects AC in E; also the distance between the angular point and the centre of gravity is two-thirds of the whole line, that is BF $= \frac{2}{3}$BE. But, by theor. II. book 3, Simp. Geom. it is 2BE$^2 + \frac{1}{2}$AC$^2 =$ AB$^2 +$ BC^2; hence BF $(= \frac{2}{3}$BE$) = \frac{2}{3}\sqrt{\dfrac{\text{AB}^2 + \text{BC}^2 - \frac{1}{4}\text{AC}^2}{2}} =$ 12·238373. In the same manner AF is $= \frac{2}{3}\sqrt{\dfrac{\text{AB}^2 + \text{AC}^2 - \frac{1}{2}\text{BC}^2}{2}} =$ 24·5854519, and CF $= \frac{2}{3}\sqrt{\dfrac{\text{AC}^2 + \text{CB}^2 - \frac{1}{2}\text{AB}^2}{2}} = 18·77350379$. H.

VI. QUESTION 97, *by Mr.* R. Tappér.

A person observed that if twice his age in years on the 16th of January 1721, were increased by its square root, and also by 5, and the result squared and added to his age, the amount would be 2256. What was his age at that time?

Answered by Mr. C. Mason.

It is evident this question is composed from that in page 225 of Ward's Mathematician's Guide.

Suppose $\dfrac{aa}{2} =$ to his age.

$aa + a + 5 = 47$. Involved is

$a^4 + 2a^3 + 12aa + 11a + 30 = 2256 = b$. Then by completing the square will be produced this canon,

$\sqrt{(\sqrt{(b + 0\cdot25)} - 5\cdot25)} - 0\cdot5 = a$, and $\frac{aa}{2} = 18$ years old.

Remark. The method of completing the square in the above original solution is thus.——Since $a^4 + 2a^3 + 12a^2 + 11a + 30$ is $= (aa + a + 5)^2 + aa + a + 5 = b$; by completing the square, we have $aa + a + 5 = \sqrt{(b + 0\cdot25)} - 0\cdot5$; and, by completing the square again, it is $a = \sqrt{(\sqrt{(b + 0\cdot25)} - 5\cdot25)} - 0\cdot5 = 6$. And then $\frac{1}{2}aa = 18$. H.

VII. QUESTION 98, *by Mr.* T. Rasberry.

Three persons A, B, and C had the misfortune to lose considerable sums of money in the grand South Sea scheme. A lost two thousand pounds more than B, and nine thousand more than C, and the square of A's loss was equal to the sum of the squares of the losses of the other two : What sum did each lose ?

Answered by Mr. Lewis Evan.

Put a, e, $u =$ the several losses, $b = 2000$, and $c = 9000$.

Then $\left\{ \begin{array}{l} a = b + c \\ e = c \\ u = b \end{array} \right\} + \sqrt{2bc} = \left\{ \begin{array}{l} 17000 \text{ A} \\ 15000 \text{ B} \\ 8000 \text{ C} \end{array} \right\}$ lost.

Additional Solution.

The theorems, in the original solution above, for determining the values of the three quantities required, are found thus :

Denoting the two differences between the greatest and each of the other two by b and c, as above, and the least quantity itself by x; then $x + c =$ the greatest, and $x + c - b =$ the middle one : hence, by the question, $(x + c)^2 = x^2 + (x + c - b)^2$; from which equation we have $x = b + \sqrt{2bc}$.

Then $x + c - b = c + \sqrt{2bc}$,
and $x + c = b + c + \sqrt{2bc}$,

which are the three quantities required. H.

THE PRIZE QUESTION, *by* Adrastea.

Wanting to know the depth of Eldon Hole, in the Peak in Derbyshire. I let a stone fall into it, and counted 8 vibrations of a pendulum 61 inches in length, from the instant the stone began to fall

to the time the sound of its striking the bottom reached the ear: Required the depth of the pit?

Answered.

August 18, 1718, Adrastea let fall 3 several stones into Eldon Hole, which fell with little or no obstruction, and accelerated in 9, 11, and 10 seconds from their delivery to the sound's reaching the ear: Sir Isaac Newton says, heavy bodies fall 16·5 foot the first second, then 3, 5, 7, 9, &c. times so much in the succeeding seconds, and that sound moves 968 feet in a second. Then $10'' \times 10'' =$ 100 × 16·5, 1650 feet ÷ by 968 = $1'' 42'''$ for the approach of sound. So the stone was falling but $8\frac{1}{4}''$; 8·25 × 8·25 = 68·0625 × 16·5 = 1123 feet = 374·34 yards.

But by Dr. Halley's numbers, let $a = 16$ foot 1 inch, $h = a \div$ 1142 = the time sound is in moving the distance a, $t =$ the time, x the depth; there will arise this theorem *,

$$x = \frac{aht + \frac{1}{2}a}{h^2} - \frac{a}{2hh} \sqrt{(4ht + 1)} = 1265·82 \text{ feet.}$$

N. B. The method taught in Lexicon Technicum, and several authors, to find a depth by the falling of heavy bodies, is false. The time the sound is reaching the ear, must be deducted out of the latter part of the stone's falling. In that author's example, under the word pendulum, according to the time of the body's falling, the well is 256 feet deep; then he allows for the sound's ascending $\frac{1}{4}$ of 16 feet to be subtracted, makes it just 250 feet: whereas in the last second, the stone fell 112 feet, therefore $\frac{1}{3}$ of that (at least) must be deducted, it will be but 225 feet.

Additional Solution.

This problem is of the nature of Sir I. Newton's 50th prob. in his Universal Arith. who has there given the true method of solution much in this manner.

Put x for the depth of the hole, $a = 16\frac{1}{12}$ feet the distance fallen by a body in the first second of time, and $b = 1142$ feet = the velocity of sound per second; also $t =$ the given time of hearing the sound, found by the experiment.——Then, by the nature of falling

---

* This theorem is wrong printed in the original, and h is put = 1142 instead of $a \div 1142$. But, as it now stands, it agrees with the result in prob. 50, Newton's Universal Arithmetic, and is derived immediately from the resolution of the quadratic equation $\frac{x}{b} + \sqrt{\frac{x}{a}} = t$ in Dr. Hutton's additional solution, by writing $\frac{a}{h}$ instead of b; for then

$$x = \frac{ath + \frac{1}{2}a}{h^2} - \frac{a}{2h^2}\sqrt{(4th + 1)} \quad \text{E.}$$

bodies, $\sqrt{a} : \sqrt{x} :: 1'' : \sqrt{\dfrac{x}{a}} =$ the time of the stone's falling to

the bottom; and by the uniform motion of sound, $b : x :: 1'' : \dfrac{x}{b}$

$=$ the time in which the sound will ascend through the space x.
Then the sum of these two times will be equal to the whole time,

that is, $\dfrac{x}{b} + \sqrt{\dfrac{x}{a}} = t$.——Now the times in which pendulums make

an equal number of vibrations, being as the roots of the lengths, and
a pendulum 39·2 inches long vibrating 8 times in 8 seconds, we shall

have $\sqrt{39\cdot2} : \sqrt{61} :: 8'' : 8\sqrt{\dfrac{61}{39\cdot2}} = 10''$ (extremely near) $= t$ the

time in which a pendulum of 61 inches swings 8 times: then the

above theorem $\dfrac{x}{b} + \sqrt{\dfrac{x}{a}} = t$ becomes $\dfrac{x}{1142} + \sqrt{\dfrac{x}{16\frac{1}{12}}} = 10$.

Hence $x = 1270\cdot4$ feet $= 423\frac{1}{2}$ yards $=$ the depth required.   H.

Questions proposed in 1723, *and answered in* 1724.

I.   QUESTION 99, *by Mr.* C. Mason.

The story of Hero and Leander has immortalized the names of
those two lovers.   Leander was a young man of Abydos, a town
on the Asiatic side of the Bosphorus or Hellespont; and Hero a
young damsel of Sestos, a town on the European side.   Leander
used to visit Hero by swimming over the straight in the night.   Now
supposing him to keep pace with the waves, which were forty feet in
breadth, how long would he be in swimming over, and what would
be the breadth of the Hellespont, the angle subtended by Abydos and
Sestos, at a station equally distant from both being 43°, and at another
station, five furlongs from the first, still equally distant from the
towns, 48°?

Answered by Mr. Rich. Tapper.

As the rectangle of the radius and the sine of 2° 30′ : the rectangle
of the sines of 24° and 21° 30′ :: 3300 = 5 furlongs : half the

breadth of the Bosphorus 11277·764613 feet. The whole breadth is 7515·1764 yards = 34 furlongs 35 yards = 4 miles 2 furlongs 35 yards. Then (according to Sir Isaac Newton) a wave runs through its breadth, whilst a pendulum, whose length is equal to that breadth, oscillates or vibrates once. A pendulum of 40 feet in length vibrates 17·14642817 times a minute. Consequently he swam 685·85 feet per minute. From hence the time is 32′ 53″ 11‴ 52⁗.

Additional Solution.

If c and D represent the two places of observation, and AB half the breadth of the Bosphorus: then we have given all the angles and the line CD, to find AB; viz. $\angle$ C = 21° 30′ = half the first observed angle, $\angle$ BDA = 24° = half the latter, $\angle$ DAC = ($\angle$ D — $\angle$ C =) 2° 30′, and CD = 5 furlongs = 3300 feet.——Hence, by common trigonometry, s. CAD : s. C :: CD : DA, and radius : s. D :: DA : AB; hence by compounding these two proportions, &c. radius × s. CAD : s. D × s. C :: CD : AB, that is, radius × s. 2° 30′ : s. 24° × s. 21° 30′ :: 3300 feet : half the breadth of the Bosphorus, as in the original solution.
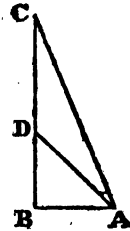
Again, by the nature of pendulums, $\sqrt{\dfrac{39·2}{12}} : \sqrt{40} :: 1'' :$

$\sqrt{\dfrac{40 \times 12}{39·2}} = \dfrac{10\sqrt{6}}{7} = 3\frac{1}{2}$ seconds, very nearly, = the time in which a pendulum of 40 feet makes one vibration, or = the time in which the wave or the swimmer moves 40 feet. And, consequently, as 40 feet : $3\frac{1}{2}''$ :: the whole distance : time of swimming. H.

II. QUESTION 100, *by Mr.* John Richards.

Find the distances from Edystone light-house to Plymouth, Start Point, and Lizard respectfully, from the following data.

The distance from $\begin{cases} \text{Plymouth to Lizard 60} \\ \text{Lizard to Start Point 70} \\ \text{Start Point to Plymouth 20} \end{cases}$ miles.

$\begin{rcases} \text{Plymouth} \\ \text{Lizard} \\ \text{Start Point} \end{rcases}$ bears from Edystone rock $\begin{cases} \text{North} \\ \text{W.S.W} \\ \text{E. by N.} \end{cases}$

Answered by Mr. John Topham.

This question is to be found in the Philosophical Transactions, proposed by Mr. Townley, with Mr. Collins's constructions to all

its cases. From Edystone to Lizard 53·112 miles, to Start Point 17·134, to Plymouth 14·187.

The question being somewhat curious, I shall here give the method of solving it. Because any two of the angles answering the rumbs given in the question, exceed 180 degrees, the point where Edystone stands must fall within the triangle. Then if you suppose a circle to pass through Lizard, Edystone, and Start, and bisect the line between Lizard and Start, producing such line through the centre of that circle to the periphery, lines drawn from the point in the periphery to Lizard and Start, with the two lines from Edystone to Lizard, and from Edystone to Start will be a trapezia inscribed in a circle, which according to 22 Euclid 3, the two opposite angles taken together are equal to two right angles ; consequently the angle at the periphery is 11° 15′, the complement to that at Edystone 168° 45′ given. In like manner let a circle pass through Plymouth, Edystone, and Lizard, the angle at the periphery will be 67° 30′, the complement to the other rumb given, 112° 30′. Again, if from the centre of the first circle you draw lines to Lizard and Start, the angle contained between them will be double to that at the circumference, by 20 prop. Euclid 3, viz. 22° 30′, and in the other circle 135°. Then,

s. 11° 15′ : log. of ½ 70 miles given :: radius : semi-diam. circle 179·4. Also s. 67° 30′ : log. of ½ 60 miles given :: radius : semi-diam. circle 32·47.

Now there is given the two semi-diameters of the circles, and the angle between them = 116·38, to find the angle subtended by the lesser semi-diameter.

As sum sides : difference :: tang. ½ opposite angles : tang. of ½ the difference of the angles ; which angle is 8° 31′. Then radius : log. 179·3 :: s. 8° 31′ : log. of 26·52 ; which doubled is the distance between Edystone and Lizard 53·04 miles. For the other two, there are two sides, and an angle opposite to one of them, given, to find the other side. Hence from Edystone to Plymouth is 14·333 miles, and from Edystone to Start Point 17·36.

First Additional Solution.

Construction.

With the three given distances, 60, 70, and 20, make the triangle LPS, wherein P will be Plymouth, L the Lizard, and s the Start. Then if E be supposed to represent the Edystone, the direction EP being north, ES being E. by N. and EL being W.S.W. there will be known all the angles about the point E. Hence, supposing PE produced to A, if on LS there be described the segment of a circle LES to contain the given angle LES = 15 points or 168° 45′, and then
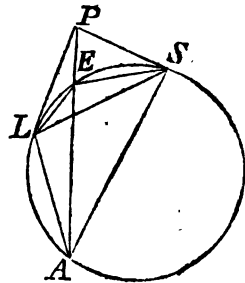
there be made either the angle SLA $= 9$ points $= \angle$AES, or the angle LSA $= 6$ points $= \angle$AEL, LA or SA cutting the circle in A; it is evident that E will be the point of intersection of AP with the circle.

Calculation.

1. In the triangle LPS, are given all the sides, and of consequence the angle PSL.——2. In the triangle ALS, are given LS, the $\angle$ASL (by constr.) and the $\angle$A $=$ the supplement of $\angle$LES; to find SA.—— 3. In the triangle PSA, are given PS, SA, and $\angle$S; to find $\angle$PAS $= \angle$SLE, and $\angle$APS.——4. In the triangle LES, are given the angles and the side LS; to find LE and ES.——5. And, lastly, in the triangle PSE, are given the angles and the side SP; to find PE.   H.

Second Additional Solution.

Let the given sides PL and PS be denoted by a and e; the given angles PES and PEL by m and n; and the unknown angles PLE and PSE by x and z. Then, by trigonometry,

$\sin m : \sin z :: e : \text{PE} = e \sin z \div \sin m; \sin n : \sin x :: a : \text{PE} = a \sin x \div \sin n;$

therefore $\dfrac{e \sin z}{\sin m} = \dfrac{a \sin x}{\sin n}$, or $\dfrac{\sin z}{\sin x} = \dfrac{a \sin m}{e \sin n}$; and therefore

$$\frac{\sin z + \sin x}{\sin z - \sin x} = \frac{a \sin m + e \sin n}{a \sin m - e \sin n}.$$

But by Theo. xxii. p. 65, vol. 3, of the Course $\dfrac{\sin z + \sin x}{\sin z - \sin x} =$

$\dfrac{\tan \frac{1}{2}(z + x)}{\tan \frac{1}{2}(z - x)}$.

Therefore $\dfrac{\tan \frac{1}{2}(z + x)}{\tan \frac{1}{2}(z - x)} = \dfrac{a \sin m + e \sin n}{a \sin m - e \sin n}$; and therefore

$\tan \frac{1}{2}(z - x) = \dfrac{a \sin m - e \sin n}{a \sin m + e \sin n} \times \tan \frac{1}{2}(z + x).$

But the sum of the angles z and x is given, being equal to the difference between the angles LES and LPS, when the point E is within the triangle LPS, as in the present case, and equal to the difference between the sum of those angles and four right angles when it is without the triangle; therefore the difference of the angles z and x is given. Now having given the sum and difference of the angles z and x, the angles themselves are given. Hence the sides PE, LE, and SE may be readily found.   L.

III. QUESTION 101, by Mr. John Willingham.

A person upon the top of a mountain observed the moon just rising one evening, and found her zenith distance to be $90° 50'$: at the same time he observed a ship at sea through a tube inclined to the

horizon 1° 58′ : required the height of the mountain and the distance of the ship from its top?

Answered by Mr. Tapper.

As radius : the secant of 50′ : : semi-diam. earth : the semi-diam. and height of the mountain in one sum. Sem. earth : s. 88° 2′ : : last sum : s. angle opposite 91° 47′. Consequently the angle at the centre is known.

If the diameter of the earth be, as Norwood makes it, 41899310 English feet, the mountain's height will be = 2216·473 feet, the ship's distance 14·6288 miles, and from the base of the mountain on the arch of a great circle 12·726.

N. B. The proposer gives this answer, viz. The mountain's height 746 yards ¼, at 70 miles to a degree; the ship's distance 3 leagues 2 miles. But some others, according to the difference of miles to a degree, 881 yards and 5 leagues, &c.

Remark. The above original solution will be evident from this figure; in which c represents the centre of the earth, AB the hill, s the ship, and AM a tangent to the earth directed to the rising moon. For, M being a right angle, and ∠ACM = 50′, whose tangent is MA and secant AC, it will be as radius : sec. ∠ACM : CM :: CA. Again, the ∠SAC being = 88° 2′, and the sides CS, CA known; we thence find the third side AS = the distance of the ship from the top of the mountain ; as also the ∠SCA, and consequently the arch SB the distance of the ship from the bottom of the mountain. H.
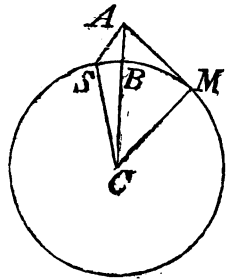
IV.   QUESTION 102, *by Mr.* John Simmons.

Wanting the dimensions of a stone, in the form of the frustum of an octagonal pyramid, which lay partly immersed in water; the greater end was just covered, but the less end only touched the water, it was known that if a line equal to the sum of the slant side and less end be divided in extreme and mean proportion, the slant side will be the greater segment of that line. The obtuse angle formed by the slant side and diameter of the circle inscribed in the less end was 100°, and the solid content of the whole frustum 158821·64 cubic inches : what are the dimensions of the frustum, and the content of the part immersed in the water?

Answered by Mr. Tapper.

Let *a* = ·82847, the area of an octagon whose diameter is 1, c = con-

tent of the frustum, $s =$ sine $10°$, $c =$ its cosine, $r =$ radius, $x =$ the diameter of the lesser base, $y =$ the slant side of the frustum more the lesser base; then $axx =$ the area of the lesser base, and the slant height $= y - x$. Therefore $y : y - x :: y - x : x$ by the nature of the line. And $yy - 2yx + xx = yx$. Hence $y = 1.5 + \sqrt{\frac{1}{4}} x$.
And $y - x = .5 + \sqrt{\frac{5}{4}} x =$ the slant side.

As radius : $.5 + \sqrt{\frac{5}{4}} x :: s : \dfrac{.5s + s \sqrt{\frac{5}{4}x}}{r} =$ the half difference of

the breadths of the two bases. And, consequently, $1 + \dfrac{s + 2s \sqrt{\frac{5}{4}x}}{r}$

$=$ the breadth of the greater base $= bx$. Then the area of the greatest base $= abbxx$, and the geometrical mean area $= abxx$. Then,

As radius : $.5 + \sqrt{\frac{5}{4}} x :: c : \dfrac{.5c + c \sqrt{\frac{5}{4}x}}{r} =$ the altitude of the

frustum $= dx$. And by multiplication we have $\frac{1}{3} adx^3 \times (bb + b + 1)$ $= c = \frac{1}{3}$ of the three areas multiplied by dx the height $= mxxx$. And hence $x = \sqrt[3]{(c \div m)} = 41.633534$ inches. Then making the square root of the area of the lesser base $= d$, the greater $=$ D, the height $=$ H.

By J. Painter.

		By J. Painter.
DD $+ \dfrac{\text{DD}d}{\text{D} + d} \times \frac{1}{3}$ H $= 107707.75$ the wet part.		107784.57
$dd + \dfrac{\text{D}dd}{\text{D} + d} \times \frac{1}{3}$ H $= 51113.89$ the dry part.		51637.07
Breadth of lesser base	41.6335	41.6641
Side of lesser base.................	17.2451	17.2530
Breadth } of greater base......... {	65.0286	65.0720
Side	26.9357	26.9524
Altitude	66.3410	66.3290
Slant height	67.3674	67.4040
Area of { lesser base	1435.9551	1437.66
greater base	3503.2256	3507.19

Additional Solution.

The solution to this question will perhaps appear a little clearer thus.—The slant side being the greater part, and the diameter of the circle inscribed in the less end the less part of a line divided in extreme and mean proportion; the meaning of which is, that the greater part is a mean proportional between the less part and the whole, that is $xx = (x + y) \times y = xy + yy$, putting $x =$ the greater part or the slant height, and $y =$ the less or the diameter of the circle in the less end. Now from this quadratic equation is found $y = \dfrac{\sqrt{5} - 1}{2} x = ax$, putting $a = \dfrac{\sqrt{5} - 1}{2}$.

Now, the slant side making an angle of $100°$ with the less end, or $80°$ with the greater, put s and c for the sine and cosine of $80°$; then sx will be the altitude of the solid, and cx the half difference

between the diameters of the two ends; and consequently $y + 2cx$ or $ax + 2cx =$ the diameter of the circle in the greater end $= bx$, putting $b = a + 2c$.—Then, putting n for the area of the octagon, the diameter of whose inscribed circle is 1, we shall have na^2x^2 and nb^2x^2 for the areas of the two ends, and $nabx^2$ their mean proportional, hence the content will be $sx \times \dfrac{na^2x^2 + nabx^2 + nb^2x^2}{3} = 158821\cdot64 = d$,

from which is found $x = \sqrt[3]{\dfrac{3d}{ns \times (aa + ab + bb)}}$. Consequently all the parts become known, and thence the content of each hoof as in the original solution. H.

<div align="center">v. QUESTION 103, <i>by Mr.</i> Will. Doidge.</div>

In a garden is a plot of ground in the form of an ellipse, whose principal axes are 15 and 10 feet; a line cuts the transverse axis at the distance of 3·2 feet from the centre, and makes therewith an angle of 72°. What are the segments of that line?

<div align="center"><i>Answered by Mr. J. Painter,</i></div>

The length of the greater part of the ordinate 5·18, and the lesser 4·23.
By Mr. *Tapper*, 5·13416 and 4·20625.

<div align="center"><i>Solution.</i></div>

Let DD be the line cutting the transverse AB in an angle of 72 degrees at the point E; draw the perpendiculars DF.

Put $m =$ semi-transverse, $n =$ semi-conjugate, $a =$ CE, s and $c =$ sine and cosine $\angle$ E $= 72°$, and $x =$ DE. Then DF $= sx$, EF $= cx$, and CF $= a \pm cx$; hence, by the nature of the ellipse, $m^2 : n^2 :: (m + a \pm cx) \times (m - a \mp cx) :$ $s^2x^2 = n^2 \times \dfrac{m^2 - (a \pm cx)^2}{mm}$; this equa-

tion reduced becomes $xx \pm \cdot9282248x = 21\cdot5948$; hence $x = 4.206$ or $5\cdot1342 =$ the two parts required. H.
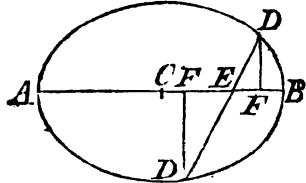
<div align="center">vi. QUESTION 104, <i>by Mr.</i> Tho. Williams.</div>

Æneas whilst contemplating with astonishment, the embellishments of rising Carthage, observed a block of polished marble, in the form of a square pyramid, weighing 48 tons. The perpendicular height was equal to twice the diagonal of the base; what were its dimensions, and how many cubic feet did it contain?

Answered.

A cubic inch of marble weighs 1·56185 ounces avoirdupois, so the content of the pyramid is 634·5729257 solid feet. The proportion of the side of a square to its diagonal, is as 1 to 1·4142. Multiply the solid content by 3, and divide the product by 2, gives 952·782, which multiplied by the square of 1·4142 = 1·99936, gives 1906, whose cube root is 12·319, the diagonal. Divide 952·78 by 12·31 gives 76·89, whose square root 8·768 is the side of the square sought.

Additional Solution.

The content being found as in the original solution above, put it $= c$, and $x =$ the side of the square base. Then $xx =$ the area of the base, $x\sqrt{2} =$ the diagonal, and $2x\sqrt{2} =$ the height. Hence the content $(2x^3 \sqrt{2}) \div 3 = c$; then $x = \sqrt[3]{(3c \div 2\sqrt{2})}$ the side of the base, and $(2x\sqrt{2} =) \sqrt[3]{24c} = 2\sqrt[3]{3c} =$ the height. H.

THE PRIZE QUESTION.

Wanting to find the height of Coventry steeple, I measured a base line, of 282·2 feet, from the bottom, along an uniform declivity, and then I could just see the top of the spire over a building; whose top was on a level with the bottom of the steeple, and distant from it 228·75 feet. It was also found that the distance from the top of the spire to the end of the base line, and the height of the steeple, were together equal to 775·04 feet. What was the height of the steeple?

Answered by Mr. J. Painter.

Put $CA + CD = 775·04 = s$, $AD = 282·2 = d$, $AB = 228·75 = b$; Substitute $2x = ss - dd$, which by the process in Ward's Math. Guide, p. 336, at the 21st step, you will have $\dfrac{ssaa - 23a^3 + a^4}{bb + aa} =$ $\dfrac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa}$. Which being reduced by multiplication and transposition, will stand, $- bba^4 - 2xa^4 + 2sbba^3 + 2xsa^3 - 2bbxaa - bbssaa - xxaa + 2bbxsa = xxbb$. In numbers, $- 573376·72a^4 + 484945075·246a^3 - 126569950746·798aa + 21131282571666·926a = 3551576300531368·689$. If the terms be divided, it will stand, $- a^4 + 845·77a^3 - 220744·848aa + 36854112·9a = 6192379528·849$.

From hence will be found a the height of the steeple $= 310·8409$ feet.
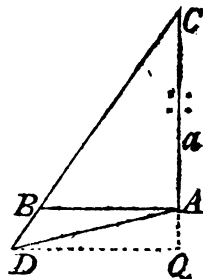
Remark. The process for finding the equation in the above original solution, may be thus :

Put $s =$ AC $+$ CD, $b =$ AB, $d =$ AD, and $z =$ AC the height of the steeple ; then $s — z =$ CD. Draw DQ perpendicular to CA produced in Q.——Now, by right-angled triangles, BC $= \sqrt{(bb + zz)}$;

and, by similar triangles, CB : CD :: $\left\{ \begin{array}{l} \text{CA : CQ} = \dfrac{s-z}{\sqrt{(bb+zz)}} z \\[2mm] \text{AB : QD} = \dfrac{s-z}{\sqrt{(bb+zz)}} b \end{array} \right\}$;

hence QA $=$ QC $-$ CA $= \dfrac{s-z}{\sqrt{(bb+zz)}} z - z$; and then, by right-angled triangles again, $ddl =$ AQ$^2 +$ QD^2, which equation reduced, and $2aa$ written for $ss — dd$, we have

$$\left. \begin{array}{c} + 2a^2 \\ + b^2 \end{array} \right\} z^4 \left. \begin{array}{c} - 2a^2s \\ - 2b^2s \end{array} \right\} z^3 \left. \begin{array}{c} + a^4 \\ + 2a^2b^2 \\ + b^2s^2 \end{array} \right\} z^2 - 2a^2b^2sz = a^4b^2, \text{ which}$$

is the equation required.   H.

———◉✖◉———

Questions proposed in 1724, and answered in 1725.

I. QUESTION 105, by Mr. Williams.

The breadth of the ditch or moat of a tower is 93 feet, and if the height of the tower be multiplied by 93, and the product added to twice the length of a ladder reaching from the outside of the ditch to the top, the sum will be 14423 : What is the height of the tower ?

Answered by Mr. John Painter.

Let $b = 93$, $c =$ ladder's length, $a =$ tower's height. Then is $ab + 2c = 14423$ per question, and $aa + bb = cc$, 47 Euc. I. There will arise this ambiguous equation $a = \frac{1}{2}d \pm \sqrt{(\frac{1}{4}dd — z)} = 151\cdot2688$ the tower's height; and the ladder's length $= 177\cdot57$ *.

Additional Solution.

The ladder being the hypothenuse of a right-angled triangle, whose base is the breadth of the moat, and perpendicular the height of the tower; if b be put $= 93$ the breadth of the moat, and $x =$ the height

* In this solution d is put for $\dfrac{2b \times 14423}{b^2 - 4}$ and z for $\dfrac{4b^2 - 14423^2}{b^2 - 4}$.

of the tower, we shall have $\sqrt{(b^2 + x^2)} =$ the ladder; and then
$bx + 2\sqrt{(b^2 + x^2)} = 14423 = c$ by the question; which equation
being reduced and a put $= b^2 - 4$, we have $x = \dfrac{bc - 2\sqrt{(ab^2 + c^2)}}{a}$.

H.

II. QUESTION 106, *by Mr.* John Richards.

There is a spheroidal cask, such, that whether the mean diameter
of an equal cylinder be found by Oughtred's rule, or by the common
method of adding $\frac{7}{10}$ths of the bung, to $\frac{3}{10}$ths of the head diameter,
the result will be the same. The length of the cask is 40·5 inches,
and when full contains 98 gallons: now supposing it to want 11
inches of being full, when measured at the bung: What quantity of
liquor has been drawn off?

Answered by Mr. John Knipe.

The bung diam. 32·1152 ⎱ inches for ⎰ 29·066 ⎱ inches for
The head diam. 23·3168 ⎰ ale gall. ⎱ 21·103 ⎰ wine gall.

The ullage wine gall. ⎰ 32·672   Ale gall. ⎰ 28·018 dry.
⎱ 65·327   ⎱ 69·981 wet.

The same answered by Mr. Rich. Tapper.

Let H = the head diameter, B = bung, D the difference B — H.

By Mr. Oughtred's theorem $\sqrt{\dfrac{2\text{BB} + \text{HH}}{3}} =$ the diameter of an
equal cylinder.

But, $\cdot7\text{B} + \cdot3\text{H} =$ the diameter of the equal cylinder per question;
by involution, transposition, and division, we have $\text{BB} - \frac{126}{53}\text{HB} =$

$-\frac{73}{53}\text{HH}$, and $\text{HB} - \frac{126}{73}\text{HB} = -\frac{53}{73}\text{BB}$. Hence $\text{B} = \frac{73}{53}\text{H}$; and H

$= \frac{53}{73}\text{B}$.

Wherefore it will be as $53 : 73 :: \text{H} : \text{B}$, and the contrary, uni-
versally: and the diameters 21·103 and 29·066 for wine gallons, or
23·316 and 32·115 for ale.

Let $x = \frac{1}{2}$ difference of the wet and dry inches, $b = \frac{1}{2}$ bung diameter.
Then $2\sqrt{(bb - xx)} =$ the breadth of the surface of the liquor in the
middle of the cask; and multiplying by $\dot{x}$, the fluent will be $2bx -$

$\dfrac{x^3}{3b} - \dfrac{x^5}{20b^3} - \dfrac{x^7}{56b^5} - \dfrac{5x^9}{576b^7} -$ &c. $=$ the area of the middle seg-

ment of a circle to the radius b, and the chord line's distance from the centre in the bung diameter x.

Now, to find the length of the radius, at any distance from the bung diameter.

Let $a = \frac{1}{2}$ length of the spheroid, $l = \frac{1}{2}$ length of the cask, $h =$ head diameter. Then $aa : bb :: (a + l) \times (a - l) = (aa - ll) : hh$, hence $aahh = aabb - llbb$; put $c = bb - hh$, then $aa = \dfrac{llbb}{c}$.

Let $z =$ the distance of the circle from the bung diameter whose radius is sought. Then $\dfrac{llbb}{c} : bb :: \dfrac{llbb - czz}{c} : \dfrac{llbb - czz}{ll} =$ the square of the radius; which raised to the $\frac{1}{2}$ power, it will give the

radius $= b - \dfrac{czz}{2bll} - \dfrac{ccz^4}{8b^3l^4} - \dfrac{cccz^6}{16b^5l^6} - \dfrac{5c^4z^8}{128b^7l^8} - \dfrac{7c^5z^{10}}{256b^9l^{10}}$ &c.

which raised to its several powers, and substituted in the room of b, it will give the area. The series multiplied by the fluxion of z, gives the solidity or quantity of liquor. Then find the fluent of each term, and let $l = z$, the series will shew $\frac{1}{4}$ of the difference of the wet and dry parts; or, if $l =$ the whole length, gives the $\frac{1}{4}$ difference, which added or subtracted with the contents, shews the quantity of liquor in the cask $65\cdot5541$, vacuity $32\cdot4458$ wine gallons.

Additional Solution.

Putting b and h for the bung and head diameters; by Oughtred's rule $\sqrt{\dfrac{2b^2 + h^2}{3}} =$ the mean diameter $=$ (by the other method mentioned in the question) $\dfrac{b - h}{10} \times 7 + h = \dfrac{7b + 3h}{10}$; from this equation is found $h = \frac{53}{73}b$; that is the bung and head diameters are exactly in proportion as 73 to 53.——Now let $73z$ and $53z$ be the the two diameters; then, supposing the content 98 ale gallons, $\dfrac{2 \times (73z)^2 + (53z)^2}{1077\cdot157} \times 40\cdot5 = 98$; hence $z = \sqrt{\dfrac{98 \times 1077\cdot157}{40\cdot5 \times (2\cdot73^2 + 53^2)}}$ $= \cdot4399361$, and consequently the two diameters $73z$ and $53z$ are $32\cdot11535$ and $23\cdot31662$.

Then for the ullage of the cask, it may be *truly* found by the rule in Cor. 2, page 290, Mensuration. Or, to use the approximating rule given at page 545 of the same $\frac{11}{32\cdot11535} = \cdot3425155$; the circular segment answering to this quotient is $\cdot23785866$; then $\cdot23785866$ $\times \frac{98}{\cdot7854} = 29\cdot68$ gallons $=$ the ullage nearly.   H.

III.　QUESTION 107, *by Mr.* A. Naughley.

At an election, four candidates offered themselves, and the whole number of votes was 5219; the number for the first candidate ex-. ceeded those for the second, third, and fourth, by 22, 73, and 130 respectively: How many had each?

Answered by Sylvia.

The sum of any whole quantity and the excesses of its greatest part above each of the other, being divided by the number of parts gives the greatest, and thereby all the other parts. Hence the first had 1361, the second 1339, the third 1288, and fourth 1231 votes.

Remark. The rule mentioned in the original solution above, is thus discovered: Let the sum s be divided into n parts, of which the differences between the greatest and each of the rest are a, b, c, d, &c. Then if x be the greatest, $x - a$, $x - b$, $x - c$, $x - d$, &c. will be the others; and their sum $x + (x - a) + (x - b) + (x - c) + (x - d)$ &c. or $nx - a - b - c - d$ &c. $= s$; hence $nx = s + m$, putting m for the sum of the differences, and $x = \dfrac{s + m}{n} = $ the greatest; which is the rule.　H.

IV.　QUESTION 108, *by Mr.* Richard Whitehead.

There is a triangular field the two shorter sides of which are $65\frac{1}{4}$ and $52\frac{89}{550}$ perches, and in the field is a rectangular space the longest side of which is $46\frac{7}{2\cdot1\cdot5}$ perches, and coincides with the third side of the field; the breadth of this space is 18 perches. Now suppose the field to be shared amongst four brothers A, B, C, D; C to have the rectangular space, B the triangular space above the rectangle, D the space on the right, and A the space on the left of the rectangle: What is the quantity of land contained in each person's share?

Answered by Mr. A. Naughley.

Let $b = 65.5$ perches, $c = 52.16$, $d = 46.98$, $p = 18$; $a, e, o, u, y,=$ several segments of the lines in the triangle. Then $ee - yy = uu$, $aa - dd + 2dy - yy = uu$, $cc - 2ce + ee - oo = pp$, by 47 Euc. I. $c : e :: b : a$, by 2 Euc. VI. and $o + y : c :: y : e$, by 4 Euc. VI.; whence this equation $\dfrac{bb}{cc} ee - ee + \dfrac{2de}{c - e} \times$ $\sqrt{(cc - pp + ce - 2ce)} = dd = 2077.9612$, which reduced and

extracted, gives $e = 28\cdot32$; consequently $a = 35\cdot36$, $o = 15\cdot63$, $y = 18\cdot56$, $u = 21\cdot39$, and the whole base $86\cdot539$.

	A.	R.	P.			A.	R.	P.
A's share ... =	1	1	16	c's share.. =		5	1	5
B's .... =	3	0	20	D's ...... =		0	3	21

Additional Solution.

The original solution may be explained by this figure, in which EFGD is the rectangle inscribed in the proposed triangle ABC, and BH perpendicular to AC; then the notation makes $b = $ AB, $c = $ BC, $d = $ FG, and $p = $ EF; also $a = $ FB, $c = $ BG, $o = $ DC, $u = $ BI, and $y = $ GI $= $ DH; and hence the equations and proportions in the solution are evident.
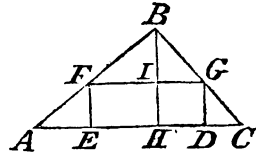
But the solution will be simpler thus;
Put b, c, d, and p, for AB, BC, FG, and GD respectively, as in the original solution, and $\dfrac{AB}{BF} = \dfrac{BC}{BG} = \dfrac{AC}{FG} = x$: and hence $dx = $ AC,

and BG $= \dfrac{c}{x}$. Then $dx : b + c :: b - c : \dfrac{bb - cc}{dx} = $ AH $-$ HC;

hence CH $= \dfrac{dx}{2} - \dfrac{bb - cc}{2dx} = \dfrac{ddxx - bb + cc}{2dx}$; and then BH$^2 = $ BC^2

$- $ CH$^2 = c^2 - \left(\dfrac{ddxx - bb + cc}{2dx}\right)^2 = \dfrac{4b^2c^2 - (d^2x^2 - m^2)^2}{4d^2x^2}$, by put-

ting m^2 for $b^2 + c^2$. Again, CG : GD :: CB : BH $= \dfrac{px}{x - 1}$. Con-

sequently $\dfrac{4b^2c^2 - (d^2x^2 - m^2)^2}{4d^2x^2} = \left(\dfrac{px}{x-1}\right)^2$; from which equa-

tion x is found $= 1\cdot85$ nearly, and hence the other quantities all become known. H.

V. QUESTION 109, by Mr. Wm. Doidge.

A father has concealed the age at which he will permit his daughter to marry in the equation $a^{\frac{3}{2}} - a^{\frac{2}{3}} = 4\cdot962a$, where a denotes the age. Pray ladies inform her at what age she may be a bride, the young lady and her lover being afraid of the time passing without their knowledge, being both unskilled in algebra?

Answered by Mr. Tapper.

Given $a^{\frac{3}{2}} - a^{\frac{2}{3}} = 4\cdot962a = ba$ to find a. Let $x^6 = a$; then $x^{18} = aaa$, and $x^{12} = aa$, also $x^{\frac{18}{2}} - x^{\frac{12}{3}} = bx^6$, which when con-

tracted is $x^9 - x^4 = bx^6$, and by transposition and division $x^3 - bx^2 = 1$. Here x will be found $= 1\cdot7425615449 +$, and $x^6 = a = 27\cdot9981125 = 27$ years, 364 days, 13 hours and $\frac{1}{2}$ *fere*, or 28 years *proxime*.

VI. QUESTION 110, *by Mr.* George Brown.

A ship at sea having been long tossed by a tempest, at length the storm abated, and, to the great joy of the crew, the sun made his appearance on the 10th of May, when his altitude was observed to be 39° 7′; and in an hour and fifty minutes afterwards, the ship lying in the same place, his altitude was found to be 55° 57′. Required the latitude of the ship, and the times at which the altitudes were taken ?

Answered by Mr. Nat. Browne.

The latitude 46° 33′, the hours 8ʰ 24ᵐ and 10ʰ 15ᵐ.

Solution.

Here are given two observed altitudes of the sun (supposed to be freed from the errors of refraction), with the (supposed) true interval of time between the observations, as also the declination of the sun ; to find the latitude and true times of observation. It is admitted that the watch or time-keeper may be too fast or two slow, but yet however it is supposed to go true for the interval of time between the observations. The declination for the supposed middle true time between the times of observation, may be used for the common declination at both times, unless the interval of time be great ; but if this be thought not sufficiently accurate, let the declination be taken for the two supposed true times themselves, and used in the calculation.

Construction.

On a proper plane, as a primitive, describe the triangle OP⊙, making ⊙P, OP equal to the co-declinations for the two times, and the angle at P equal to the given interval of time ; then will P be the pole, and ⊙, o the places of the sun at the first and second observations. Again, about the poles ⊙, o describe two circles at the distance of the two observed co-altitudes and intersecting in z the zenith of the place. Then a great circle described through P, z will be the meridian, the arc PZ = the co-latitude, and the angles ZP⊙, ZPO the measures of the times required.
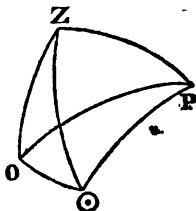
Calculation.

Drawing the great circles, zo, z⊙. In the triangle op⊙ are given the two sides op, ⊙p equal the two co-declinations, and the included ∠op⊙ = the interval of time; to find o⊙ and the ∠o⊙p. Then in the triangle oz⊙, are known the three sides ; to find the ∠o⊙z ; which taken from the ∠o⊙p, there remains the ∠z⊙p. Lastly, in the triangle z⊙p, are known the two sides z⊙, ⊙p, and included ∠z⊙p ; to find the side pz the latitude, and the ∠zp⊙ the time for the first observation from noon ; from which taking the given difference op⊙, there remains the time of the second observation from noon. H.

<center>THE PRIZE QUESTION, *by* Adrastea.</center>

Two military roads, called by the Romans Watling-Street, and the Fosse, intersected each other at a place formerly named Benones, now called High-Cross. Dove-Bridge or Tripontium, bears nearly south-east by south, from High-Cross, and distant from it $8\frac{9}{64}$ miles : at Dove-Bridge the angle subtended by High-Cross, and another place upon the Fosse, called Eathorpe, is a right angle. Brinklow-Hill is another place upon the Fosse, between High-Cross and Eathorpe, and distant from the latter place $8\frac{151}{160}$ miles ; and at Brink-low-Hill the line to Dove-Bridge is perpendicular to the Fosse. The above data are sufficient for determining the distances of Brinklow-Hill and High-Cross, and of Eathorpe and Dove-Bridge. At a place called Newbold, the angle subtended by High-Cross and Dove-Bridge was 88° 15' ; and the angle subtended by Dove-Bridge and Eathorpe 149° 45'. Required the distance of Newbold from each of the places, High-Cross, Brinklow-hill, Eathorpe, and Dove-Bridge ?

<center>*Answered.*</center>

In the first part of the question there is given in a right-angled triangle ABC, the cathetus AB, and the alternate segment of the hypothenuse CD, (made by a perpendicular BD, let fall from the right angle) ; to find the other segment DA. Put $c =$ AB, $b =$ CD, and there will arise this theorem, $a = \sqrt{(cc + \frac{1}{4}bb)} - \frac{1}{2}b = 4.81903$, and by 47 Euc. I. the base $= 11.0836$, the perpendicular DB $= 6.55$; which completes the triangle.

Then from E where Newbold stands, there is given the angle AEB 88° 15' ; BEC 149° 45' ; and CEA 122°. To find the dis-

tances EA, EB, EC, and ED. And here we must note, as in the answer to question 100, that as any two of the angles given, together exceed 180 degrees, the point where Newbold stands will fall within the triangle ABC.
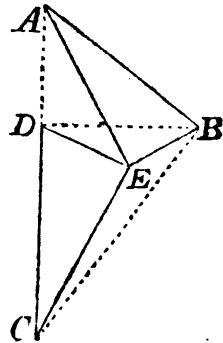
Now if you suppose a circle to pass through c Eathorpe, E Newbold, and B Tripontium, and bisect the line CB, producing that bisection through the centre of the circle, it will cut the periphery in a point from whence lines drawn to B and C, will complete a trapezia inscribed in a circle; which by 22 Euc. III. has the two opposite angles taken together equal to two right angles : hence the angle at the periphery will be 30° 15′. In like manner proceed with the 3 points AEB, and the angle at the periphery will be 91° 45′.

Then if from the centre of the first circle you draw lines to B and c, the angle contained between them will, by 20 Euc. III. be double to that at the circumference, viz. 60° 30′, and in the other circle it will be 183° 30′.

But it will be needless to explain the matter any farther, having already, in the last year's diary, given as clear a demonstration as could be, without a large scheme that would admit of all those lines and circles requisite thereto ; what remains being purely trigonometrical, from whence I have deduced this following answer.

			M.	F.	P.	Tapper. M.	F.	P.
Given {	From High-Cross to Tripontium	AB	8	1	5	8	1	5
	From Eathorpe to Brinklow ..	CD	8	7	22	8	7	22
	To complete the triangle ...							
	Brinklow to High-Cross	DA	4	6	22	4	6	21
	Tripontium to Eathorpe.......	BC	11	0	32	11	0	29
	Brinklow to Tripontium.......	DB	6	4	16	6	4	20
	Eathorpe to Benones	CA	13	6	4	13	6	3
	High-Cross	EA	7	4	24	7	4	26
Newbold to {	Tripontium	EB	3	2	2	3	2	1
	Eathorpe	EC	8	1	16	8	1	11
	Brinklow......	ED	4	1	16	4	1	20

Mr. Rich. Tapper *answers in the manner following.*

Let $a =$ the distance of Tripontium and Eathorpe; $c =$ the distance of Tripontium and High-Cross $= 2605$ poles ; $n =$ the distance of Brinklow from Eathorpe $= 2862$ poles ; then $a = \sqrt{((ccnn + \frac{1}{4}n^4)^2} - \frac{1}{4}nn) = 3549.9109$ poles ; which call d. Let $s =$ sine of 30° 15′; m its cosine, or sine of 149° 45′; $z =$ the sine of 88° 15′, x its cosine, $1 =$ radius.

Imagine the line which joins Newbold and Tripontium to be extended through the Fosse, and a perpendicular let fall from Eathorpe upon it; also a perpendicular from High-Cross upon the said line. Then $d : s :: a : as \div d =$ the sine of the angle made by the lines which join Tripontium and Newbold, and Tripontium and Eathorpe,

$1 : d :: as \div d : as =$ the length of the first perpendicular, $1 : d$ $:: m : ma =$ the distance from Newbold to the place where the said perpendicular cuts the extended line; and $\sqrt{(dd - aass - ma)} =$ the distance of Newbold from Tripontium, by 47 Euc. I. Now $\dfrac{\sqrt{(dd - aass)}}{d} =$ the cosine of $\dfrac{as}{d}$. Then, $z : c :: \dfrac{\sqrt{(dd - aass)}}{d}$

$: \dfrac{c}{zd} \sqrt{(dd - aass)} =$ the distance from Newbold to High-Cross; $1 :$

$\dfrac{as}{d} :: c : \dfrac{cas}{d} =$ the distance from Tripontium to the place where

the perpendicular from High-Cross would cut the line between New-

bold and Tripontium, and $1 : c :: \dfrac{\sqrt{(dd - ssaa)}}{d} : \dfrac{c}{d}\sqrt{(dd-ssaa)}$

$=$ the perp. also $z : \dfrac{c}{d} \sqrt{(dd - ssaa)} :: x : \dfrac{xc}{dz} \sqrt{(dd - ssaa)} =$ the

distance of the perpendicular from Newbold. Hence this equation,

$1 - \dfrac{xc}{dz} \times \sqrt{(dd - ssaa)} = ma + \dfrac{sca}{d}$, which reduced gives this

theorem $a = \dfrac{d - \dfrac{cx}{z}}{\sqrt{\left\{ \left(\dfrac{cs}{d} + m\right)^2 + \left(s - \dfrac{scx}{dz}\right)^2 \right\}}} = 2612 \cdot 7383$ poles.

Additional Solution.

In the first place we have given, in a right-angled triangle, one leg AB, and the alternate segment CD of the hypothenuse made by the perpendicular BD; and the triangle will be constructed as question 28, page 24. Again, the point E, where three lines AE, BE, CE make given angles with each other, will be determined as in question 100, page 127. H. .

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Questions proposed in 1725, and answered in 1726.

I. QUESTION 111, *by Mr.* Rich. Whitehead.

Suppose D, C and G to be the respective positions of a house, a tree and a boat on the borders of a lake, the distances of which from

one another are to be determined. For this purpose the following lines were measured, *viz.*

DB $= 147\frac{1}{4}$, the prolongation of a line joining D and C

BA $= 474$ a perpendicular drawn to DC, meeting CG produced in A

AG $= 579$ the prolongation of CG

GF $= 674\frac{1}{3}$ the prolongation of a line drawn from D to G.

FE $= 507\frac{1}{3}$ a perpendicular let fall from F on DC produced

CE $= 259$ the distance of the perpendicular from C.

Hence it is required to find DC, DG, and GC?

Answered by Mr. C. Mason.

Let	1	BD $= b = 147\frac{1}{4}$
	2	AB $= c = 475$
	3	AG $= d = 579$
	4	FG $= f = 674\frac{1}{3}$
	5	EF $= g = 507\frac{1}{3}$
and	6	CE $= h = 259$
put	7	DF $= a$

By E. 6.   8 $a : g :: a - f : \dfrac{ga - gf}{a} = $ GE

9   $c - \dfrac{ga - gf}{a} : d :: c : \dfrac{cda}{ca - ga + gf} = $ AC,

for $c - g$ put x

9 In. 2   10 $\dfrac{ccddaa}{xxaa + 2xagf + ggff} - cc = $ BC^2

10 — 1   11 $\sqrt{\left(\dfrac{ccddaa}{xxaa + 2xagf + ggff} - cc\right)} - b = $ DC

12   $\sqrt{(aa - gg)} - h = $ DC

13   $\sqrt{\left(\dfrac{ccddaa}{xxaa + 2xagf + ggff} - cc\right)} - b = \sqrt{(aa - gg)} - h.$
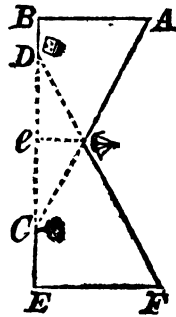
Which equation reduced, and the sums collected, &c. gives $a = 1078\cdot0833$: Then $a - f$ is the distance between the house and boat DG $= 403\cdot7499$, and DC the distance between the house and tree $= 692\cdot2499$, and GC between the boat and tree $= 386$.   *Q. E. I.*

N. B. *The letter* G *is omitted at the boat in the preceding figure.*

Additional Solution.

The final equation will be simpler by substituting x for the unknown

perpendicular Ge, the known quantities being as in the original solution ; for then, by similar triangles,

$$g - x \; : \; g \; :: \; f \; : \; \frac{fg}{g - x} = \text{DF},$$

and $\quad c - x \; : \; c \; :: \; d \; : \; \frac{cd}{c - x} = \text{AC} ;$

hence, by right-angled triangles and subtraction,

$$\text{DC} = \sqrt{\left(\frac{ccdd}{(c-x)^2} - c^2\right)} - b, \text{ and } \text{DC} = \sqrt{\left(\frac{ffgg}{(g-x)^2} - g^2\right)} - h ;$$

$$\sqrt{\left(\frac{ccdd}{(c-x)^2} - c^2\right)} - b = \sqrt{\left(\frac{ffgg}{(g-x)^2} - g^2\right)} - h.$$

And here if CE or h be supposed equal to BD or b, as those distances are arbitrary, and might as well be measured equal as unequal, then the final equation will become barely $\dfrac{ccdd}{(c-x)^2} - c^2 = \dfrac{f.fgg}{(g-x)^2} - g^2,$ in which x is easily found.   H.

II.   QUESTION 112, *by Mr.* Tho. Dodd.

From what height above the surface of the earth must a heavy body fall, that the time of its descent may be equal to the time of ascent of sound to the same height, the velocity of which is known to be uniform and 1142 feet per second.

Answered by Mr. Sam. Rouse.

Sound moves 1142 feet per second ; heavy bodies fall $16\frac{1}{12}$ feet in the first second, according to Dr. Halley. Then put $a = 16\frac{1}{12},$ $b = 1'',$ $x =$ height, $d =$ the time sound flies $16\frac{1}{12}$ feet ; $a : x ::$ $bb \; : \; \dfrac{bbx}{a} =$ square of the time of the descent of the body. And $a : x :: d : \dfrac{dx}{a} =$ the time sound moved to the height x. Consequently $\dfrac{dx}{a} = \sqrt{\dfrac{xbb}{a}}$ whence $\dfrac{bbx}{a} = \dfrac{ddxx}{aa}, \; \therefore x = \dfrac{bba}{dd} =$ 81088·085 feet $=$ 27029·36 yards $=$ 15 miles, 2 furlongs, 34 perches 7 feet.

Additional Solution.

The solution will be a little clearer thus : Putting $a = 16\frac{1}{12},$ $c =$ 1142, and $x =$ the height.   Then $\sqrt{a} : \sqrt{x} :: 1'' : \sqrt{\dfrac{x}{a}} =$ time

of falling x feet; and $c : x :: 1'' : \dfrac{x}{c} =$ time of sound's passing

over x feet; therefore $\dfrac{xx}{cc} = \dfrac{x}{a}$, and $x = \dfrac{cc}{a} = \dfrac{1142^2}{16\frac{1}{12}} = 81088$

feet.   H.

III.   QUESTION 113, by Mr. Tho. Grant.

A father at his death left an estate of 5000l. per annum to his three daughters to be shared in the proportion of their several ages: now the square of each age added to the product of the other two, will make the three sums 1000, 980, and 920; required their ages and fortunes?

Answered by Mr. John Turner.

Let $\begin{cases} aa + eu = b = 1000 \\ ee + au = c = 980 \\ uu + ae = d = 920 \end{cases}$ by the question, then

$$\left. \begin{array}{l} + 8u^8 - 18du^6 + 2bcu^4 - b^3uu + bbcc \\ \qquad + 18ddu^4 - c^3uu - 2bcd^2 \\ \qquad\qquad - bcuu + d^4 \\ \qquad + 5bcduu \\ \qquad - 7dduu \end{array} \right\} = 0.$$

Which being collected and reduced gives the following numbers.

Ages.		Portions.		
		l.	s.	d.f.
The eldest 23·5276		1784	19	0¼·7568
Second 22·7788	cash per annum	1728	2	8½·3738
Youngest 19·5991		1486	18	2¾·8694
C. Mason		5000	0	0 0

Additional Solution.

$\begin{cases} aa + eu = b \\ ee + au = c \\ uu + ae = d \end{cases}$ by the question,

or putting $e = ma$ and $u = na$

$$aa(1 + mn) = b, \text{ or } aa = \frac{b}{1 + nm},$$

$$aa(mn + n) = c, \text{ or } aa = \frac{c}{mm + n},$$

$$aa(nn + m) = d, \text{ or } aa = \frac{d}{nn + m}.$$

Whence $\dfrac{b}{1 + nm} = \dfrac{c}{mm + n}$ and $\dfrac{d}{nn + m} = \dfrac{c}{mm + n}.$

or $bmm + bn = c + cnm$ and $dmm + dn = cnn + cm.$

From the first of these equations $n = \dfrac{bmm - c}{cm - b}$, which being substituted in the second gives

$$dmm + \frac{dbmm - dc}{cm - b} = c \left(\frac{bmm - c}{cm - b} \right)^2 + cm.$$

And, by reduction,

$(bb - cd)m^4 + (bd + cc)m^3 - 4bcm^2 + (bb + dc)m + (c^2 - db) = 0,$
or in numbers, and dividing by 400,

$$246m^4 + 4701m^3 - 9800m^2 + 4754m + 101 = 0;$$

whence $m = \cdot 9683,\ n = \dfrac{bmm - c}{cm - b} = \cdot 833,$ and the values of a, e

and u are found the same as above nearly.

This problem is the same (except the numbers) as one proposed to Dr. Wallis, in the year 1662, by Colonel Titus, to whom it had been originally proposed by Dr. John Pell, the famous algebraist of that time. Dr. Wallis inserted a solution of it in his Algebra, but it is exceedingly tedious and laborious, and leads to an equation of the 12th power; it reduces, however, by division, to one of the 8th power.

Mr. Baron Maseres has inserted it in a volume of Tracts on the resolution of algebraic equations published in 1800, and has set forth every step of the solution in the fullest and clearest manner possible. Mr. Maseres has also inserted another solution by Mr. Frend, which is much simpler than Dr. Wallis's and produces only a biquadratic equation. It is similar to one given by Ronayne, in his Algebra, and is founded on the same principle as that which we have given above. There is likewise a very ingenious solution of Col. Titus's problem, by Mr. Ivory, in the 3d vol. of the first series of the Mathematical Repository, which also leads to an equation of the fourth power, but the coefficients are smaller, and the equation easier to resolve than Mr. Frend's, but it is only adapted to Col. Titus's numbers. This problem is also resolved by Kirkby in his Arithmetic. L.

IV.   QUESTION 114, *by Mr. Christ. Mason.*

What would a body equal to ten pounds troy at the earth's surface, weigh, if carried to the surface of each of the planets?

Answered by *Mr.* Rich. Whitehead.

Suppose the sun's parallax 10 seconds, and the magnitude of the earth according to Norwood's observations; then let $a =$ the distance of the earth from the sun, $p =$ the periodical revolution of the earth, $s =$ the periodical revolution of one of $\mathcal{4}$ satellites. To find its distance from $\mathcal{4}$ say $p^2 : s^2 :: a^3 : \dfrac{a^3 s^2}{pp} =$ the cube of the distance sought.

And as the squares of these periods recip. so are the centrip. forces towards their respective central bodies.

Then for the weight of bodies on their surfaces, if the distances are equal, the weights will be as the quantity of matter the bodies contain. If the quantities of matter are equal, the weights will be reciprocally as the squares of the distances : *Ergo*, If neither are equal, the weight will be in a compound ratio. For, as the distance between the sun and the earth, and the quantity of matter in the earth, is to ten pounds, so is the distance of Jupiter, and quantity of matter, to the weight ten pounds would weigh on his surface; and so for the rest. Hence ten pounds of our troy weight will weigh on the surface of the sun 244 pounds, on Jupiter 20, on Saturn 12·75, on the empire of the bedlamites, *viz.* the moon 3·45.

Mr. Alexander Naughley's *answer.*

On the surface of the sun 240, $\mathcal{4}$ 19·9, $\male$ 6, $\leftmoon$ 5·15, $\mercury$ 2·15, $\venus$ 2·08, $\saturn$ 17.

Mr. *Tho.* Dod computes them from

	$\astrosun$	$\mathcal{4}$	$\saturn$	$\leftmoon$
Mr. Whiston's numbers	245	19·5	13·0	3·2
And Dr. Cheyne's theory	79·3	6·4	4·2	5

Additional Solution.

The proportions of the force of gravity at the surfaces of the planets, or the proportions of the weights of an equal quantity of matter at each, are now stated thus:

Sun	Moon	Mercury	Venus	Earth	Mars	Jupiter	Saturn
25	$\frac{1}{4}$	$\frac{1}{95}$	$\frac{4}{5}$	1	$\frac{1}{3}$	2	$1\frac{1}{3}$

These being severally multiplied by 10, we have

250	$3\frac{1}{3}$	$\frac{2}{19}$	8	10	$3\frac{1}{3}$	20	$13\frac{1}{3}$

for the weights required. H.

v. QUESTION 115, *by Mr.* Bernard Annely.

On the 1st of June, 1724, at 55 minutes past seven in the even-

ing, it then wanting $7\frac{1}{2}$ minutes of sun-set, a person observed the sun in a direct line with the top of a lofty building, 97 feet high, which stood upon the same level with himself : What was the distance of the building ?

Answered by Mr. John Turner.

The time was 7^h. $55'$, and ⊙ setting in that latitude was 8^h 2^m 30^s. The longitude Ⅱ $21°$ $50'$ $17''$. Declination $23°$ $13'$ $53''$. Ascensional difference $30°$ $37'$ $30''$. To find the latitude, having the declination and ascensional difference, tan. declination $23°$ $13'$: radius :: s. ascensional difference $30°$ $37'$: tan. lat. $49°$ $52'$ $53''$. Then there are given two sides and an angle included, to find the third side the cosine of the sun's altitude $57'$ $52''$. Then s. ⊙ altitude : height of the object :: cosine of sun's altitude : length of the shadow $5762\cdot4$ feet $= 1920$ yards.

Additional Solution.

The time 8^h $2\frac{1}{2}^m$ of sun-set is found by adding $7\frac{1}{2}^m$ the half quarter to 7^h 55^m the time when the sun is directly behind the top of the edifice; at which time let o denote the sun's place, and ⊙ the point of sun-set; also let z be the zenith, and p the pole; and let the great circles be drawn as in the figure.
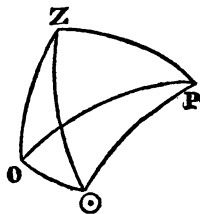
Then will zpo be the hour angle of 7^h 55^m and zp⊙ the hour angle of 8^h $2\frac{1}{2}^m$ p⊙ or po the given co-declination, pz the co-latitude, zo the co-altitude at the time of observation, and z⊙ the co-altitude at the time of setting, which will be a quadrant, it being the distance between the zenith and horizon.

Wherefore, in the triangle zp⊙ are given z⊙, p⊙, and the $\angle$ zp⊙ ; to find zp the co-latitude. Then in the triangle zpo, will be given zp, po, and their included $\angle$ zpo; to find zo the co-altitude at the time of observation. And, lastly, the distance will be equal to the length of the shadow, which is the base of a right-angled plane triangle whose perpendicular is the height of the building, and the angle at the base the sun's altitude; whence appears the reason of the last stating in the original solution. H.

VI.   QUESTION 116, by Mr. John Simmons.

There is a piece of land in the form of a rectangle, the sum of whose perimeter and diagonal is just half a mile: and it was remarked by a surveyor who had measured it, that if a right line equal

to the sum of the diagonal and less side, was divided in extreme and mean proportion, the greater segment would be equal to the diagonal; What are the sides of the rectangle, and what will it come to at $1\frac{1}{4}$d. a square yard?

Answered by Mr. Whitehead.

Let $a =$ oblong's greater side, $e =$ lesser, $u =$ diagonal, and $b = 880$ yards $= \frac{1}{2}$ mile, in proceeding several steps you will have $\sqrt{5} \times u = b - 2a$; put $g = \sqrt{5}$; then completing the square $aa + ka + \dfrac{kk}{4} = \dfrac{bbg - bb}{14 - 4g} + \dfrac{bbgg - 2bbg + bb}{49 - 28g + 4gg} =$

$\dfrac{45bbg - 10bbgg - 35bb}{880 - 588g + 168gg - 16g^3} = \dfrac{12098496 \cdot 879012673 +}{32 \cdot 30659103014048} =$ 374490·04968136201.

$a + \dfrac{k}{2}$ (i. e. $a + \dfrac{bg - b}{7 - 2g} = 430 \cdot 299969) = 611 \cdot 9559213549$.

Hence the greater side of the oblong is $a = 181 \cdot 65595$; e the lesser $= 142 \cdot 80907$; diagonal $= 231 \cdot 06994$. Content $25942 \cdot 1189$ yards at $1\frac{1}{4}$d. comes to $135l$. $2s$. $3\frac{1}{2}594$d.

Additional Solution.

If $z =$ the diagonal, and $x =$ the breadth; then $xz + xx = zz$, and hence $x = z \times \dfrac{\sqrt{5} - 1}{2}$. Also, by right-angled triangles, the length $= \sqrt{(zz - xx)} = z \sqrt{\left(1 - \left(\dfrac{\sqrt{5}-1}{2}\right)^2\right)} = z \sqrt{\dfrac{\sqrt{5}-1}{2}}$.

So that the breadth, length, and diagonal, are to each other as $\dfrac{\sqrt{5}-1}{2}$, $\sqrt{\dfrac{\sqrt{5}-1}{2}}$, and 1; and therefore are in continual proportion.

Now the sum of this diagonal with twice the length and breadth is $\sqrt{5} + \sqrt{(2\sqrt{5} - 2)}$; and then, by proportion, as this sum is to half a mile (the sum given in the question), so is each of the above proportional quantities to the breadth, length, and diagonal of the figure required. Wherefore half a mile or 880 yards drawn into each of the fractions $\dfrac{\dfrac{\sqrt{5}-1}{2}}{\sqrt{5} + \sqrt{(2\sqrt{5}-2)}}$, $\dfrac{\sqrt{\dfrac{\sqrt{5}-1}{2}}}{\sqrt{5} + \sqrt{(2\sqrt{5}-2)}}$,

and $\dfrac{1}{\sqrt{5} + \sqrt{(2\sqrt{5} - 2)}}$, will produce the required breadth, length, and diagonal.   H.

THE PRIZE QUESTION.

When the Romans under Marcellus besieged Syracuse we are told that Archimedes contributed greatly to the defence of the place by the invention of curious and powerful warlike engines. Let us suppose that one of them, for throwing large stones, was placed upon the city wall, to annoy the nearest point of the Roman camp, and that the distance of the camp from the ditch' at this place was 260 yards; also that at a corner of the wall, distant from the engine 400 yards, Archimedes stood viewing the point assailed, and found the distance across the ditch in the direction of that point to be 200 yards. What distance would he have stood from the point assailed?

Answered by Mr. Sam. Marriot.

In this right-angled triangle wherein a line is drawn parallel to the base, there is given the base $= 400$; that segment of the hypo-thenuse next the base $200 = c$; and the alter-nate segment of the cathetus $260 = d$. To find the cathetus, &c.
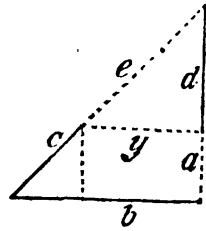
$$d + a : b :: d : \frac{db}{d + a} = y,$$

$$e + c : b :: e : \frac{be}{e + c} = y = \frac{bd}{d + a} \quad \text{per}$$

sim. triangles. Hence $bde + bea = bdc + bed$, or $bdc = bea$, and $\frac{dc}{a} = e$. Again, $bb + dd + 2da + aa = \frac{ccaa + 2dcca + ccdd}{aa}$ per 47 Euc. I.

Hence $a^4 + 2da^3 + (bb + dd - cc) aa - 2ccda = ccdd.$

Then $a = 141\cdot73$; $a + d = 401\cdot73$ the distance sought; $c + e$ from first stat. to the camp $566\cdot908$; $y =$ length of the ditch $258\cdot883$ yards.

Mr. R. Tapper *answers it thus:*

Let $a =$ the distance of the wall from the ditch, $b = 260$ yards, $n = 200$, $m = 400$. $\left(1 + \frac{b}{a}\right) + \sqrt{(nn - aa)} = m$, which reduced will be

$a^4 + 2ba^3 + (bb + mm - nn) aa - 2bnna = bbnn.$

Here a will be found 141·7005; $a + b = 401·7005$ the distance of the camp of Marcellus from Syracuse.

The solution of this question may more fully be seen in Ward's Introduction, p. 334.

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Questions proposed in 1726, and answered in 1727.

I. QUESTION 117, *by Mr.* Richard Whitehead.

A company of mathematicians, after emptying an elegant glass punch bowl, found themselves quite at leisure to contemplate its figure, &c. The inside had the form of an hyperbolic conoid, the transverse diameter of the generating hyperbola being 6·93 inches, and the conjugate 5·29 inches; the form of the outside of the bowl was that of a cone generated by the revolution of the asymptote of the hyperbola, and the length of the outside was $8\frac{1}{3}\frac{2}{8}\frac{7}{6}\frac{8}{6}\frac{8}{5}$ inches; also the depth of the bowl was 5·98 inches. Required the internal diameter at the brim, weight of the bowl, and what liquor it contained when full?

Answered by Mr. Christ. Mason.

$cc = b = 9·445$, D$s = c = 2·645$, AD
$= d = 8·777$, $cs = a = ?$ $b - a = cs$

$a : c :: b : \dfrac{bc}{a} = $ CB, $b - a : a :: d :$

$\dfrac{da}{b - a} = $ DC, $a : \dfrac{da}{b - a} :: b : \dfrac{bd}{b - a}$

$= $ AC, $\dfrac{bbcc}{aa} = $ CBq; $\dfrac{bbdd}{bb - 2ba + aa}$

$= $ ACq, $\dfrac{bbdd}{bb - 2ba + aa} - bb = $ CB$q = \dfrac{bbcc}{aa}$.
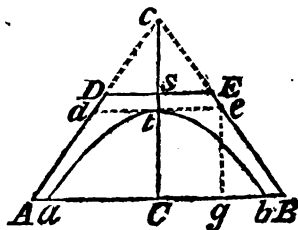
Which equation reduced, &c. gives $a = 2·93$, from which all the other dimensions may be easily found. Then *per Archim. de Conoid & Spheriod, Def.* 3, the truncated cone AdeBb*ta* will be $=$ a cylinder generated by the parallelogram c*teg.* Ergo, from the frustum AdeBA, take that cylinder, the remainder will be $=$ the solidity of the hyperbolic conoid *atba*; to which cylinder add the frustum d*D*E*ed*, the

K 4

sum will be equal the solidity of the bowl $=$ 197·873 inches. The solidity of the conoid $=$ 499·774 inches $=$ 2·1635 wine gallons.

Then by $\begin{Bmatrix} \text{Phil. Trans.} \\ \text{Ward,} \end{Bmatrix}$ $\begin{Bmatrix} \text{an inch of} \\ \text{glass weighs} \end{Bmatrix}$ $\begin{Bmatrix} 1\cdot54282 \\ 1\cdot493037 \end{Bmatrix}$ ounces avoird.

And the weight $\begin{Bmatrix} (1) & 305 & \text{oz.} = 19\text{lb. 1oz.} \\ (2) & 294\frac{1}{4} & \text{oz.} = 18\text{lb. 7oz.} \end{Bmatrix}$ avoirdupois. of the bowl

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II.   QUESTION 118, *by Mr.* Christ. Mason.

Divide 10,000 into two such parts, that when each is divided by the other, the sum of the quotients will make 5.

Answered by Mr. R. Whitehead.

The two numbers $\begin{Bmatrix} 1726\cdot73164 \\ 8273\cdot26835 \end{Bmatrix}$ added make 4·99999 &c.

Solution.

Let x and y denote the two numbers, s their sum, and q the sum of their alternate quotients.——Then $x + y = s$, and $\dfrac{x}{y} + \dfrac{y}{x}$ $= q$; Hence $xx + yy = qxy$, and $xx + 2xy + yy = ss$; or, by substitution, $qxy + 2xy = ss$; therefore $4xy = \dfrac{4ss}{q+2}$; and by subtraction, $xx - 2xy + yy = ss - \dfrac{4ss}{q+2} = \dfrac{q-2}{q+2} \times ss$; there‑ fore $x - y = s\sqrt{\dfrac{q-2}{q+2}}$.

Consequently $\begin{cases} x = \frac{1}{2}s + \frac{1}{2}s\sqrt{\dfrac{q-2}{q+2}}, \\ y = \frac{1}{2}s - \frac{1}{2}s\sqrt{\dfrac{q-2}{q+2}} \end{cases}$ .... H.

III.   QUESTION 119, *by Mr.* John Turner.

In surveying an elliptical inclosure, three lines of 200, 313·62, and 870·62 feet, were measured from one of the foci to the curve; the first and second made an angle of 25° with each other, and the second and third an angle of 65°. Required the area; also the length of a ditch, which cuts the transverse axis 200 feet from the centre, the lesser segment of the ditch being the least line that can possibly be drawn from the point of intersection to the curve?

Answered by Mr. W. Gill.

Dr. Keil, in his Astro. Lect. gives a method of determining an ellipse from 3 lines given in length and position, cutting each other in the focus.

The transverse diameter 1000 feet, conjugate 600, the content of the field 10 a. 3 r. 10 perches, the angle of inclination of the transverse 64° 21'. From the doctrine of fluxions, if the lesser segment be supposed to be the hypothenuse of a right-angled triangle, also a minimum, and a perpendicular let fall upon the transverse diameter, the base of the said triangle will be a fourth proportional to the transverse, the latus rectum, and the distance of the perpendicular from the centre of the ellipsis, that is $1000 : 360 :: 200 + x : x = 112\cdot5$. Hence the lesser part of the ditch $= 259\cdot8$, greater $= 330\cdot66$, and the whole length $= 590\cdot46$.

Additional Solution.

It is demonstrated, by the writers on conics, that if c be the centre and f the focus of an ellipse, and in the transverse axe BA produced there be taken CD a third proportional to CF and CA, and from any point G in the curve GE be drawn parallel to BD and meeting DE perpendicular to it in E, and FG be drawn; then FG will always be to GE, in the constant ratio of CF to CA.—Wherefore, if G, H, I be the three given points, or ends of the given lines FG, FH, FI, drawn from the focus F; and there be drawn HG, IH, and produced to K and L so that HK be to GK as FH to FG, and IL to HL as FI to FH; and then if KL be drawn, and perpendicular to it IN, HM, GE, FD; and lastly FD be divided in A in the given ratio

of FG to GE; then A will be the end of the transverse, and the whole of the ellipse is thence determined.——For, KH being to KG as FH to FG by the construction, and KH to KG as MH to EG by similar triangles, therefore FH will be to FG as MH to EG, or FH to HM as FG to GE; and in the same manner FI : IN :: FH : HM; wherefore in general FI : IN :: FH : HM :: FG : GE :: (by the above cited property) CF : CA; whence the centre c and the whole ellipse becomes known.
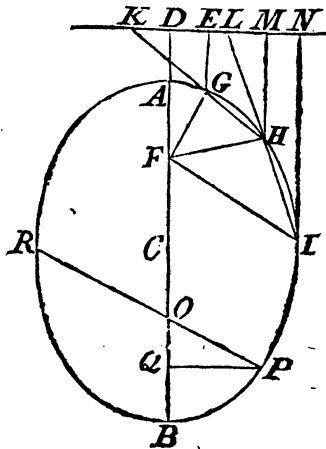
Again, for the shortest line OP that can be drawn from the given point o in the axe to the curve; since it must evidently be perpendicular to the curve or to the tangent in the point P, it may be determined

sum will be equal the solidity of the bowl $= 197 \cdot 873$ in⁰
solidity of the conoid $= 499 \cdot 774$ inches $= 2 \cdot 1635$ **wine**

Then by $\begin{Bmatrix} \text{Phil. Trans.} \\ \text{Ward,} \end{Bmatrix}$ an inch of $\begin{Bmatrix} 1 \cdot 54282 \\ 1 \cdot 493037 \end{Bmatrix}$ oun-

And the weight $\begin{Bmatrix} (1) \ 305 \ \text{oz.} = 19\text{lb. } 1\text{oz.} \\ (2) \ 294\frac{1}{4} \ \text{oz.} = 18\text{lb. } 7\text{oz.} \end{Bmatrix}$ a

II. QUESTION 118, by Mr. C

Divide 10,000 into two such parts, th
the other, the sum of the quotients wil

Answered by Mr

The two numbers $\begin{Bmatrix} 1726 \cdot 731 \\ 8273 \cdot 26 \end{Bmatrix}$

quantity

the same?

Let x and y denot

sum of their altern·

$= q$: Hence xx

substitution,

subtraction

fore x

		Mason.
₃2° 56′		As sine of the altitude **52° 11′**
37 4		Is to the height 120 inches 49
23 25	add	So cosine altitude ...... **37** in.
60 29	altitude ..	To length of shadow 93·1 **55**
sun's altitude	60 29	From which subtract 10 in ch·
height object	120 in.	Remainder 83·155.
cos. sun's alt. ...	29° 31′	
length shadow	67·989	As cosine sun's altitude **37°** 49
		To sine time past noon 31 15
As radius		So cosine ⊙'s declin... 66 34
cos. time from noon	58° 45′	To sine sun's azimuth 50· 55
cos. lat.	52 56	
tang. of a 4th arch	32 51	As radius 90 0
		Is to .... 83·155
Which sub. from ⊙'s dist. of pole		So cosine azimuth .... 50° 55″
leaves a 5th arch $= 33° 44′$		To the ordinate $ea = 52·4$
Cosine 4th arch	57 09	As radius
To cosine 5th arch ...	56 16	To .... 83·155
So sine of the lat.	52 55	So sine .... 50° 55′
To sine of the altitude	52 11	To de .... 64·53

Let $ea = b = 52\cdot4, dc = c = 67\cdot989$
$de = d = 64\cdot53$ and $et = a.$ Quære a ?
$dd + 2da + aa : cc :: da$

$$: \frac{ccdd}{dd + 2da + aa} = bb.$$

Reduced, $bbaa + 2bbda - ccda = bbdd.$ In numbers produces $a = 36\cdot86$. Then $1 :$ $\cdot7854 :: \text{T}t \times cc : 21640\cdot4668$ inches $=$ the area, by which divide the given content, gives the depth $40\cdot35$ inches. Add twice the thickness of the lead to each diameter, and once to the depth : from which elliptical cylinder subtract the given content, gives the inches of lead 11614, which according to Ward's specific gravity of lead, is equal to 76116 ounces avoirdupois $= 2$ ton, 2 hund. 1 quar. $25\frac{1}{4}$lb. weight.

Remark. The spherical calculations, in the original solution above, will be evident by applying the given declination, latitude, and hour to the spheric triangle in page 119 ; for if zP represent the co-latitude, P⊙ the co-declination, and their included angle P the hour from noon, all which are given ; then z⊙ will be the co-altitude, and the $\angle$z the azimuth, both which are hence easily found.

Again, for the ellipse, instead of the *jet d'eau* suppose a pole of 10 feet high erected in the centre d of the ellipse ; then will the shadow of it at noon be the semi-conjugate axe dc, and 10 inches less than the shadow of it when the azimuth is equal to the $\angle adc$ will be the semi-diameter $db = da$. Which are easily determined as in the original solution.

Every thing else in the original solution is very well determined except the transverse axe, which may be better found thus, by the common property of the ellipse, as $\sqrt{(dc^2 - ae^2)} : de :: dc : d\text{T}$ the semi-transverse. H.
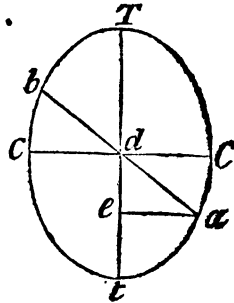
V. QUESTION 121, *by Mr.* Tho. Grant.

Not far from the city of Cairo, in Egypt, is a small round temple, whose outside is curiously ornamented with hieroglyphical figures, and encompassed by a spiral tube, sloping from the bottom to the top, in an angle of $61°\ 5'$. The circumference of the temple is $100\cdot7$ yards, and it is known that a perfectly smooth heavy ball would descend through the tube in 8 seconds. Required its height ?

Answered by Mr. Whitehead.

The proposer has either unlimited or unhinged the question, by giving the circumference of the tower incoherent with the time ; either alone would have limited it.

According to Gallilæus, and experiments made by several ingeni-

ous artists of late, as $1 : 16\frac{1}{12} :: 8'' \times 8''=64 : 1029\frac{1}{3}$ = the space
a body will describe a perpendicular descent in $8''$ of time. Then,

As the sine of given angle ACB $= 61° 5'$
Is to the side.......... AB $= 1029\frac{1}{3}$
So is the radius
To the hypothenuse .... AC $= 1175·81$

Then let fall the perpendicular aB, and while a body
accellerates the space AB, another will describe the
space Aa, along the inclined plane AC, as has been
demonstrated by the illustrious Sir Isaac Newton.

Having found the segment A$a = \dfrac{AB^2}{AC} = 901$, &c.

Say, as radius : A$a = 901$, the length of the
tube :: s. of the given angle A$ab = 61° 5' :$ A$b = 788·72$ feet, the
height required.
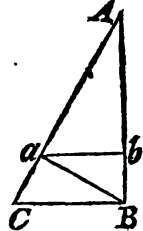
*Mr. Grant's solution to the 121st question, taken from the Diary for
1729, and who thinks Mr. Whitehead's solution was not right.*

As $1 : 193 : 64 :: 12352$ inches = the perpendicular descent of
the body by gravity in $8''$ of time: Then as the radius : sine of
$61° 5' :: 12352 : 10812$ inches = the space described in $8''$ by a
body descending along a plane that makes an angle of $61° 5'$ with
the horizon, which would be the tube's length were it a right line,
but being a spiral to the temple which the question supposes a cy-
linder whose periphery is $100·7$ yards ; it is evident, that as the ball
descends, it will also, by the tube's curvature, be carried in the said
periphery, therefore it will be acted on by a *vis centralis*, in a di-
rection parallel to the horizon, which will not forward the ball's
descent, but will constantly diminish it, and consequently the velo-
city that would arise therefrom, which will occasion the ratio of the
spaces described by the ball descending in the tube, to the times of
their description to be always variable. Wherefore, I compare in a
ratio, as the semi-ordinates of a conic parabola to their respective
abscissas, whence the following analysis : where $x =$ tube's length
the ball runs in $8''$.

Radius : cos $61° 5' :: x : ·483537x =$ to an arch described in $8''$,
in the periphery, which squared, divided by the diameter, gives
$·0002026167xx =$ the space in $8''$, agitated only by the central
force; which multiply by x^2, and make it the abscissa of a parabola
convex to its axis, x its ordinate, and the parameter will come out
$= 4935·427$ inches $= a$, the parabolic curve, by compounding the
motions along the ordinate and abscissa, will be $= 10812$ inches
= the space in $8''$, by a body descending along the inclined plane.
Then, by the doctrine of fluxions, $x = 6175$ inches $= 514\frac{7}{12}$ feet
the tube's length, and radius : s. $61° 5' :: 6175 : 5405$ inches $= 450$
feet, the temple's height.

Remark. After all Mr. Grant's labour and time spent in the
solution of this question, Mr. Whitehead's solution is right. For the

side of the tube, by impelling the ball in the horizontal direction, does not alter its perpendicular velocity or descent; and therefore 'the spiral may be considered as a straight inclined plane in the solution.   н.

VI.   QUESTION 122, *by Mr.* Sam. Marriot.

A person was possessed of a rectangular close, the fences of which had been destroyed, and the only mark left was an oak tree in the east corner; he however recollected the following particulars of the dimensions.   It had once been resolved to divide the close into two by a hedge diagonally, and he recollected that a segment of the diagonal intercepted by a perpendicular from one of the corners was 16 chains, and the same perpendicular produced 2 chains, met the other side of the close.   Now the owner has bequeathed it to four grand children, whose shares are to be bounded by the diagonal and perpendicular produced.   What are the several shares ?

Answered by Mr. Sam. Rouse.

Draw DA.   Then $\triangle$ DAB $=$ $\triangle$ FAB by Eucl. I. 37, $\therefore$ $\triangle$ DAC $=$ $\triangle$ FBC.

Put AC $= b = 2$, CF $= a$, CD $= 16 =$ d, CB $=e$.
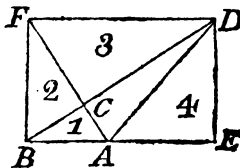
E. VI. 8.	1	$b : e :: e : a$
$\therefore$	2	$ba = ee$
But	3	$ae = bd$
$3 \div e$	4	$a = \dfrac{bd}{e}$
$2 \div b$	5	$a = \dfrac{ee}{b}$
4 and 5	6	$\dfrac{bd}{e} = \dfrac{ee}{b}$
6 ×	7	$dbb = eee$
7 w 3	8	$\sqrt[3]{(bbd)} = e = 4$ chains
E. 8. 6.	9	AC : CB :: CB : CF $= 8$ ch.

A. R. P.

(1) BCA $= 0$ 1 24 ⎫
(2) BFC $= 1$ 2 16 ⎬ shar.
(3) CFD $= 6$ 1 24 ⎪
(4) ACDE $= 7$ 2 16 ⎭

Total 16 0   0

*This question is in page* 367 Hill's Arithmetic.

THE PRIZE QUESTION.

A person travelling towards York, discovered, by means of a telescope, the top of the tower of St. Peter's Cathedral, when just in the horizon; and upon approaching 20 miles nearer, he found the top of the tower elevated one degree.   What is its height, that of the observer's eye being 5 feet 6 inches ?

Answered.

$$\text{Let} \left\{ \begin{array}{c} \\ \end{array} \right. \begin{array}{c|l} 1 & r = \text{radius of the earth} = tc, \\ 2 & h = bd \text{ or } ef = \text{height of eye}; \\ 3 & 2\,rh + hh = bt^2 \\ 4 & \sqrt{(2rh + hh)} = bt \\ 5 & be - bt = tc, \\ 6 & \angle cea = 91° \text{ per quest.} \end{array}$$

36 E. 3.

3w2

Then find the angle *tce*, and line *te*; find the angles *tec*, and *etc*; subtract the angle *etc* from 90°, it leaves the angle *eta*. Again, to the angle *tec* add $\angle cea$, which sum subtract from 360, leaves the angle *tea*. Then the angles *tca* and *ate* subtracted from 180, leaves the angle *tae*. Then,

As sine *tae* : *te* :: sine *tea* : *ta*. And $\sqrt{(tc^2 + ta^2)} - tc = ae$ $= 260\frac{3}{4}$ feet, the height sought; which is something more than the real height of the tower of York Cathedral.
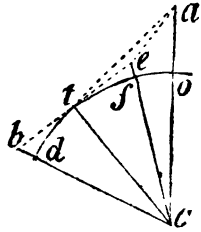
Questions proposed in 1727, *and answered in* 1728.

1. QUESTION 123, *by Mr.* Richard Whitehead.

A surveyor put the following question to an impertinent fellow who had interrupted him when busy. Suppose four towns B, C, D, and L, the distance from L to each of the other towns being as follows: BL $= 12848$, CL $= 11440$, and DL $= 7744$ yards; the $\angle$B $= 65°$ 10′·96766, $\angle$C $= 77°$ 0′·07237, $\angle$D $= 37°$ 49′. Required the distances BC, BD and DC?

Answered by the proposer's method.

Put $a = $ BD, $b = $ LB, $d = $ DL, $c = $ CL, $m = $ sine $\angle$CDB $= 6131·369$, $n = $ sine $\angle$PCD $= 7899·767$, $r = $ radius $= 10000$, $s = $ sine BCD $= 9743·748329$, $t = $ sine $\angle$CBD $= 9076·46563$.
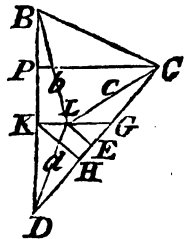
By trigonometry,

$$\frac{ta}{s} = \text{CD}, \quad \frac{mta}{rs} = \text{CP}, \quad \text{and} \quad \frac{nta}{rs} = \text{PD}.$$

By oblique triangles,

$$\frac{aa + dd - bb}{2a} = \text{KD}, \quad \frac{ttaa + sscc - ssdd}{2sta} = \text{CE},$$

and $\dfrac{ttaa + ssdd - sscc}{2sta} = $ ED.

By similar triangles,

$$PD : PC :: KD : KG = \frac{maa + mdd - mbb}{2na},$$

$$PD : DC :: KD : DG = \frac{raa + rdd - rbb}{2na},$$

$$CD : CP :: GK : GH = \frac{mmaa + mmdd - mmbb}{2rna},$$

$$CD : DP :: KD : DH = \frac{naa + ndd - nbb}{2ra};$$

hence $GD - ED = GE =$

$$\frac{rstaa + rstdd - rstbb - nttaa - nssdd + nsscc}{2nsta},$$

Then, by similar triangles,

$$CP : PD :: GE : EL = rstaa + \&c. \text{ divided by } 2msta,$$

or $EL = \dfrac{21538697327713058aa - 11 \cdot 839151262}{a} = \dfrac{qaa - p}{a}$;

also $CE = \dfrac{46575841908309173 \cdot 3aa + 12 \cdot 28641236}{a} = \dfrac{kaa + l}{a}$,

therefore $LE^2 + CE^2 = LC^2$;[1]

that is, $\dfrac{qqa^4 - 2qpaa + pp + kka^4 + klaa + ll}{aa} = cc$,

or $2pqaa + ccaa - 2klaa - qqa^4 - kka^4 = ll + pp$;

that is, by division and substitution, $xaa - a^4 = z$,

or $xe - ee = z$, by putting $e = aa$;

hence $e = aa = \dfrac{x}{2} + \sqrt{\left(\dfrac{xx}{4} - z\right)} = 127 \cdot 6958908986685$,

and $a = 11 \cdot 300260066 = BD$, and the rest

$$\left.\begin{array}{l} CD = 10 \cdot 526383 \\ CB = 7 \cdot 1108241 \end{array}\right\} \text{ miles} = \left\{\begin{array}{ccc} 11 \text{ m.} & 2 \text{ f.} & 16 \text{ p.} \\ 10 & 4 & 8 \\ 7 & 0 & 35 \end{array}\right.$$

Mr. *John Turner* has very curiously wrought this answer by trigonometry, and an algebraical process to this biquadratic affected equation,

$$(1 + g^2)x^4 + 2dxx = \begin{array}{l} - 4cc \\ - dd \\ - hh \end{array}$$
$$+ 2gh$$

$x = \sqrt{((ss - t) + s)} = 11 \cdot 2983$ miles, $10 \cdot 525$, and $7 \cdot 11$.

Additional Solution.

Describe the triangle B*cd* similar to the proposed one, or whose angles shall be equal to the given ones: then, by prob. 31, Simpson's Geometry, determine the point *l* such that the three lines B*l*, *cl*, *dl* may be in proportion to each other as the three given distances BL, CL, DL: upon B*l* take BL = the given distance of B from L; and draw LD, LC parallel to *ld*, *lc*; and join D, C: so shall B, C, B, L be the four points required.

Which is too evident to need a formal demonstration. H.
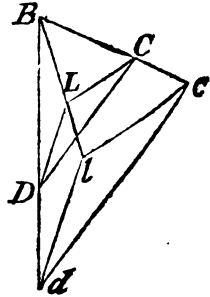
Another Additional Solution.

In the triangle BCD there are given all the angles and the lengths of the lines BL, DL, and CL drawn from the angles to a point within the figure to construct the triangle. Suppose it done. Draw BE, LE to make the triangle BLE similar to the triangle BCD, namely, the angle LBE = DBC, the angle BLE = BDC, and the angle BEL = BCD, and join EC; 'then BD : BC :: BL : BE, and because the angle LBE = DBC, the triangles BLD, BEC are similar (Euc. VI. 6); therefore BD : BC :: DL : EC, or BL : BE :: DL : EC: whence we have this construction.
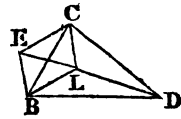
On the given line BL describe a triangle LBE, similar to the required one, and take EC a fourth proportional to BL, BE, and LD; then with the distances EC, LC and centres E, L describe two arcs to intersect in C. Make the angle BLD = BEC, and take LD of the given length; join BC, BD, and CD, and the triangle will be constructed.

For the lines BL, CL, and DL are evidently of the given lengths: and since the angle BLD = BEC and BL : BE :: DL : EC, or BL : DL :: BE : EC, the triangles BLD and BEC are similar; therefore the angle DBL = CBE, and adding the common angle CBL, the angle DBC = LBE. Also BD : BL :: BC : BE, or BD : BC :: BL : BE, and consequently the triangle BDC is similar to BLE.

The calculation is extremely easy, for in the triangle BLE we have the side BL and all the angles to find BE and LE; then in the triangle LCE all the sides are known to find the angle CEL, and then the whole angle BEC becomes known, and we have two sides and the included angle to find BC. Now all the angles of the triangle BCD being known, the sides CD and BD are easily found. J. L.

II. QUESTION 124, *by* Philosophicus.

Suppose a point to move upon the spherical surface of the earth from a place in latitude 48°, with the following compound motion, viz. along a meridian with the uniform velocity of 14 miles per hour, whilst at the same time the meridian revolves about its axis with the uniform angular velocity of 15° per hour.   Required the latitude and difference of longitude of another point in the curve so described, that the intercepted part may be of a given length ; suppose 4503 miles ?

Answered by Mr. Turner.

Let SP = 4503 miles = d, b = 14, and c = 1039 the number of miles which every point of the equator is carried through in the space of an hour, by the earth's diurnal revolution ; then the ratio of PL to SL, will be ba to ca : But $b^2a^2 + c^2a^2 = dd$, Euc. 47. 1.
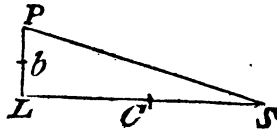
Hence $a = \sqrt{\dfrac{dd}{cc + bb}}$, consequently PL = ba = 60·676 miles, the difference of latitude, and the latitude, arrived in will be 46° 59′¼ north.

This may likewise be solved trigonometrically, and has some affinity to current sailing, the motion of the earth representing the current's motion, the point being carried by the compound motion from P to s : but each point of the parallel of 48° does not run through 1039 miles per hour, and the line PS will indeed be a spiral line upon the globe.

Mr. *Geo. Brown* has well answered and demonstrated this question, and found the latitude to be 46° 16′ : but the operation and scheme my room will not admit.

Additional Solution.

The original solution to this question is not right, because the motion in the parallel is made constantly equal to that at the equator ; whereas the intent of the question seems to be, to find the spiral described on the globe by a point, which is urged with a given constant velocity in the direction of the meridian, and a given constant angular velocity around the pole in the direction perpendicular to the meridian ; the rates being 14 miles on the meridian to the 24th part of the parallel of latitude.—Or it may be conceived by supposing a point to move along the brass meridian of a globe at the rate of 14 miles while the globe turns round 15 degrees, the motion answering to an hour ; for then the parts of the globe passing directly under the point, will trace out the spiral.

Wherefore, if x be put for the cosine of the variable latitude, and consequently $\sqrt{(1 - xx)}$ its sine to the radius 1 ; also $a = 14$ miles,

and $b = $ 1-15th part of the equator : since $bx = $ the motion per hour in the parallel of latitude, and $\dfrac{\dot{x}}{\sqrt{(1-xx)}} = $ the fluxion of the arch of the meridian or abscissa of the spiral, we shall have as $a : bx :: $ $\dfrac{\dot{x}}{\sqrt{(1-xx)}} : \dfrac{bx\dot{x}}{a\sqrt{(1-xx)}} = $ the fluxion of the parallel or ordinate of the spiral; then the sum of the squares of these fluxions of the ordinate and abscissa will be equal to the square of the fluxion of the curve of the spiral, or fluxion of the spiral $\dot{z} = \dot{x}\sqrt{\dfrac{aa + bbxx}{aa - aaxx}}$,

or $= \dot{x}\sqrt{\dfrac{1 + qxx}{1 - xx}}$ by putting q for $\dfrac{bb}{aa}$. Now, by Cor. 2, p. 228,

Mensuration, it will be found that this expression is the fluxion of an elliptic arc whose semi-axes are 1 and $\sqrt{\dfrac{aa + bb}{aa}}$. Wherefore if

the circle whose radius is 1 be circumscribed by the ellipse whose transverse semi-axe AC is $=$ $\sqrt{\dfrac{aa + bb}{aa}}$, and BD be the degrees or arc of the latitude, and EDF be drawn parallel to AC; then CF will be $= x$, and the arc AE $= z$ the above fluent, or length of the spiral from the pole: Also Ee will be the length of the spiral between any two latitudes BD, Bd. So that if BD be the given latitude of 48°, and DE be drawn parallel to AC; then take E$e = $ 4503 miles the proposed length of the spiral, and draw ed parallel to ED; so shall Bd be the latitude required.　H.
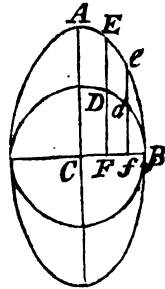
III.　QUESTION 125, *by Mr.* John Turner.

Suppose three lofty poles standing perpendicular to the horizon, at points denoted by A, L, c; the height of the pole at A being 32 feet, that at L 96, and that at c 53 feet, and the distance of A and L 200 feet. Now suppose that upon a certain day the shadow of the top of the pole at A passed through the points c and L; and the shadow of the top of the pole at L passed through A and c; and the shadow of the top of the pole at c passed through A. Upon what day, and in what northern latitude might this happen ?

Answered by Mr. Brown.

There is a demonstration of this in Sir Isaac Newton's Universal Arithmetic, too large to insert, by which I make the latitude 81° 30′, and sun's declination 70° 22′.

Mr. *Mason*, Mr. *Whitehead*, and some others say this question is unlimited, or at least admits of more true answers than one; but having received none but the above, I shall not here give my judgment till further trial be made at a more leisure time, except Mr. *Bent*'s latitude 80° 45′, declination 19° 27′.

Remark. This question has evidently been made from prob. 55. Sir Isaac Newton's Arithmetic, where a full solution may be seen. н.

IV. QUESTION 126, *by Mr*. Tho. Dodd.

It is required to dispose the under written 16 numbers into the form of a magic square, so that the sum of each horizontal, vertical, and diagonal row may amount 74.

The numbers to be disposed in the square are 8, 9, 10, 11, 14, 15, 16, 17, 20, 21, 22, 23, 26, 27, 28, 29.

Answered.

Mr. *Elias Colbourn* says, if the numbers as given be placed in four rows, you need only let the diagonals stand, and cross places with the other numbers as in these squares:

8	9	10	11
14	15	16	17
20	21	22	23
26	27	28	29

8	28	27	11
23	15	16	20
17	21	22	14
26	10	9	29

The numbers plac'd as given in the question.   Transpos'd, which makes 74 ten ways, viz. laterally, transversely, and diagonally.

Aspatia says, the method of filling all sorts of magical squares with the magic cubes, may be seen at large in the memoirs of the Royal Academy of Sciences for 1710, page 124, by Mons. Saurier, in his Construction Generale des Quarrés Magiques.

See also question 61, page 73.

V. QUESTION 127, *by Mr*. Tho. Williams.

Our modern nat'ralists dispute—Whence came
The water that destroy'd the earth's first frame?
Some bring it from the moon ; and some have thought
That from above, the firmament, 'twas brought :

Others—that 'twas created for that end,
Or from a deep abyss did then ascend.
The last the sacred writ doth intimate,
As Woodward has explain'd it to's of late:
The which abyss (with Halley) he wou'd have
Concentric with the earth, an orb concave.

    If so—an alternative change suppose;
Let air subside, whilst water earth o'erflows;
And while the earth's a sea, let air in place
Of water fill up the orbicular space.
For present use (with Burnet) we'll agree,
That the antidiluvian earth might be
Of mathematical rotundity;
And that th' abyss did hold this weight* in air,
Which we'll admit no heavy'er there than here.
From hence, fair ladies, these two things unfold:
How deep the flood; how much th' abyss might hold?

        * 1000000000000000 tons avoird.

Answered, by Mr. Whitehead.

    According to Mr. Boyle, the weight of water is to that of air as 1000 to 1. Mr. Ward makes a cubic inch of water to weigh 578697 ounces avoirdupois. Mr. Norwood makes a degree of the great circle $=69\frac{1}{4}$ miles; hence the diameter of the earth is $=504606222$ inches fere. And thence 358400000000000000000 ounces of air will fill the abyss; and thence 3584019......ounces of water will fill the same. Then as 578697 ounces : 1 inch :: 35840, &c. ounces : 61932237423038308475765 $=$ number of inches in the abyss, which added to those which are in the earth, the sum will be equal to the solid inches in the sphere of the flooded earth: from the diameter of which take that of the earth, and divide the remainder by 2, the quotient is 77397·7752 inches $=6449·8146$ feet $=2149·9382$ yards $=1$ mile 389 yards.

    This answer agrees pretty near with the proposer of the question. There were many others who solved it: but in answers to questions of this nature, there will be some considerable difference, by reason the proportions are taken from authors which do not exactly agree; as (1) In the proportion of air to water, (2) In the weight of water, (3) The geometrical miles in a degree, (4) in the diameter to the circumference.

VI. QUESTION 128, by Mr. Christ, Mason,
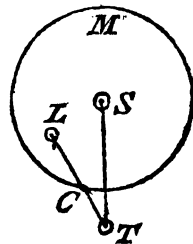
    Given the quantities of matter in the earth and moon, together

with the distance of their centres, to determine their common centre of gravity.

<div align="center">Answered by Mr. T. Williams.</div>

The common centre of gravity of the earth and moon goes through the magnis orbis so, that the areas described by the radii to the sun are proportional to the times ; the accelerating gravities are as the quantities of matter in those bodies ; the mass of the earth is given equal to 39008956283823202513476 cubic feet; the diameter of the ☽ to the diameter of the ⊖, as her apparent semi-diameter to her horizontal parallax. Hence her magnitude 801682492916446916225 feet: The attractive force will be the same in respect to their distance from the common centre of gravity, as in respect of the whole distance between them.

Let s denote the sun, мc the magnis orbis describ'd by the common centre of gravity of the earth т, and the moon ʟ, c the ☽ in octant, her mean distance $60\frac{2}{3}$ semi-diameters $= 1309095$ 060 feet $=$ тʟ. Now as the ☽ and ⊖ tend to c as to one another, cт : cʟ :: mass in ʟ : mass in т.
Therefore cт $= 26361759$ feet $= 4083$ miles, and cʟ $= 1282733200$ feet $= 242752$ miles.

Mr. *Mason*, the proposer, answers thus : $d =$ mean diameter $= 268923$ miles, $q =$ the quantity of matter in the earth, and $1 = $ ☽, $a =$ distance of the common centre from the sun ; $q : 1 :: $ mass ⊖ : mass ☽, $qa = d - a$ per statics, and $qa + a = d$; hence $a = \dfrac{d}{q+1}$ $= 5899{\cdot}34$ miles.

Mr. *Turner*, has curiously explain'd this, and says the author ought to have told whether in ☌ or ☍, &c. but that nothing certain can be laid down, since we cannot truly determine the distance and magnitnde of the sun ; and that regard should be had to their densities, and specific gravities, &c.

<div align="center">Additional Solution.</div>

If e be the quantity of matter in the earth, and m that in the moon, or any other two bodies, and d their distance asunder. Then, per statics, as $e + m : d$

$$:: \begin{cases} m : \dfrac{dm}{e+m} = \text{the distance of the earth} \\ e : \dfrac{de}{e+m} = \text{the distance of the moon} \end{cases} \text{from the common}$$

centre of gravity. н.

VII.   QUESTION 129, *by Mr.* Sam. Rouse.

A golden candlestick is suspended by a chain from the roof of a lofty building; If put in motion it would make 1000 vibrations in an hour; it is also known that the distance from the roof to the candlestick, is a mean proportional between the height of the roof, and the height of the candlestick above the floor. What is the height of the building on the inside?

Answered by Mr. Da. Nairne.

The pendulum vibrates 1000 per hour $= 16\frac{2}{3}$ per minute, and one of 39·2 inches vibrates 60 per minute; also they being in a duplicate ratio of their vibrations, therefore the length of the candlestick will be found 508 inches; and the candlestick above the floor 314. Hence the whole height $= 822$ inches $= 68\cdot5$ feet.

Additional Solution.

As $1000^2 : 3600^2$ or as $10^2 : 36^2 :: 39\cdot2 : 39\cdot2 \times 3\cdot6^2 = 508$ inches the length of the cord or line.

Again, if $x =$ the whole height, and $a = 508 =$ the greater part. Then $x : a :: a : x - a$; hence $xx - ax = aa$, and $x = \dfrac{1 + \sqrt{5}}{2}$ $\times a = 821\cdot96$ inches $= 68\frac{1}{2}$ feet nearly.   H.

THE PRIZE QUESTION.

In a beautiful garden is a circular grass plot whose radius is 50 feet; from the centre two rectilinear walks branch off, forming an angle of 55°; their lengths are 108 and 80 feet respectively. It is required to find a point in the circumference, to which lines being drawn from the extremities of the walks, they shall form equal angles with the tangent at the point of concourse.

Answered.

Join AB, AC and Am; draw ms and Bw $\perp$ AC, mq $\parallel$ AC, mf $\parallel$ AB; draw md so that $\angle mdA = \angle Amc = Amb$, and mr that $\angle Arm = Amb$. Then the triangles Amb, Amr are similar, as also Amc and Amd: also the triangles rmq, dmf are similar. Then let $a = 108 = AB$, $b = 80 = AC$, $c = 61\cdot95 = Aw$, $d = 88\cdot47 = Bw$, and $r = 50 = Am$; also $x = As$, $y = ms$. Then $mf = \frac{ay}{d}$, and $sf = \frac{cy}{d}$, which taken from x, we have $Af = x - \frac{cy}{d} = mq$, and $Aq = mf = \frac{ay}{d}$. Again,

in the like triangles, Amb, Amr, $Ar = \frac{rr}{a}$; and in Amc, Adm, $Ad = \frac{rr}{b}$, which taken from $Af = x - \frac{cy}{d}$ we have $df = x - \frac{cy}{d} - \frac{rr}{b}$.

Also $rq = \frac{ay}{d} - \frac{rr}{a}$. Then $mq : qr :: mf : df$, that is $x - \frac{cy}{d} : \frac{ay}{d} - \frac{rr}{a} :: \frac{ay}{d} : x - \frac{cy}{d} - \frac{rr}{b}$; and, crossing extremes and means, $xx - \frac{2cyx}{d} - \frac{rrx}{b} + \frac{ccyy}{dd} + \frac{rrcy}{db} = \frac{aayy}{dd} - \frac{rry}{d}$; or for

$aa - cc$ put dd, and transposing $\frac{ccyy}{dd}$, &c. comes $x^2 - \frac{2cyx}{d} + - \frac{rrx}{b} = yy - \frac{rrcy}{db} - \frac{rry}{d}$. Proceeding finds $x = 45\cdot8068 = As$.

Then the angle sAm may be found; and two sides and a contained angle given in each to find $Bm = 70\cdot32$, and $cm = 39\cdot63$ feet the answer. *I. T.*
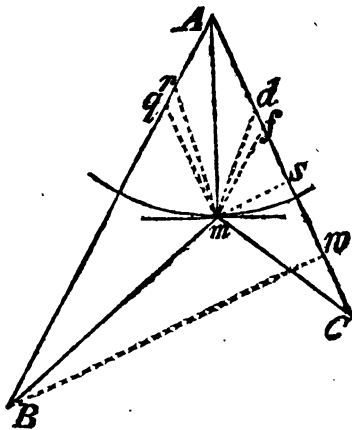
Additional Solution.

This question may be otherwise solved thus :
Putting $a = AB$, $b = AC$, and $r = Am$, as above; and also s and e = sine and cos. $\angle BAC$,

s and $\sqrt{(1 - x^2)}$ = sine and cos. $\angle BAm$, and
z and $\sqrt{(1 - z^2)}$ = sine and cos. $\angle cAm$, to the radius 1.

Then, by trigonometry, $\dfrac{ax}{r - a \sqrt{(1 - x^2)}} =$ tang. $\angle$ AmB, and

$\dfrac{bz}{r - a \sqrt{(1 - z^2)}} =$ tang. $\angle$ Amc; but these angles are equal to each other by the question; therefore we have this equation,

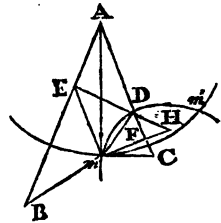$$\frac{ax}{r - a\sqrt{(1 - x^4)}} = \frac{bz}{r - b\sqrt{(1 - z^2)}}.$$

But, again by trigonometry, $z = s \sqrt{(1 - x^2)} - cx,$
  and $\sqrt{(1 - z^2)} = sx + c \sqrt{(1 - x^2)}$;

these values of z and $\sqrt{(1 - z^2)}$ being substituted in the above equation, we have $\dfrac{ax}{r - a \sqrt{(1 - x^2)}} = \dfrac{bs \sqrt{(1 - x^2)} - bcx}{r - bsx - bc \sqrt{(1-x^2)}}$; which equation, reduced, gives

$$4a^2b^2x^4 - 4a^2brsx^3 + a^2r^2x^2 + 2a^2brsx + a^2b^2s^2 = 0 ;$$
$$+ b^2r^2 \qquad - 2ab^2crs - b^2r^2s^2$$
$$- 4a^2b^2$$
$$+ 2abcr^2$$

from which biqaudratic equation x may be found.   H.

Another Additional Solution.

In this question it is required to find a point m in the circumference of a circle, given in magnitude and position, so that the two lines drawn from it to the given points B and c may make equal angles with a tangent to the circle at m. Now let A be the centre of the circle, and draw BA, CA in which let the points E and D be so taken that the rectangles BAE, CAD may be each equal to the square of the radius Am; and join Em, Dm and ED, and draw mH parallel to BC meeting CA in F and ED in H.   Then because BA : Am :: Am : AE, and CA : Am :: Am : AD, the triangle AmB is similar to AEm, and the triangle Amc to ADm, therefore the angle AmB $=$ AEm and the angle Amc $=$ ADm : but by hypothesis the angle AmB $=$ Amc, therefore the angle AEm $=$ ADm. And since the rectangle BAE $=$ CAD, the points E, D, B, c are in a circle, therefore the angle AED $=$ ACB $=$ AFm, and consequently the angle DEm $=$ AEm $-$ AED $=$ ADm $-$ AFm $=$ DmF ; therefore the triangles HmE and HDm are similar, and HE : Hm :: Hm : HD, which is the property of the equilateral hyperbola whose diameter is DE : Hence if an equilateral hyperbola be described to the diameter DE and whose ordinates are parallel to BC it will cut the circle in the point m required, and likewise in another point which will also answer the question.   And moreover if the opposite hyperbola be described

it will also intersect the circle in two other points, either of which will answer ; so that there are four points, in the circumference of the given circle, to which lines may be drawn from the given points so as to make equal angles with the circle at the point of concourse.

When the given points are equally distant from the centre of the circle the hyperbolas degenerate into two right lines, the one passing through the points E, D and the other through the centre perpendicular to ED.

It may be remarked that this question is the same as the famous problem of Alhazén relating to the reflection of light; namely, to find the ray which, issuing from a given point, shall be reflected by a a spherical surface to the eye in any given position. Which problem has, at different times, engaged the attention of some of the most eminent mathematicians of the 16th and 17th centuries, as Slusius, Huyghens, Barrow, L'Hospital, R. Simson, Robins, &c. Their solutions are all effected by means of the hyperbola, and the greater part of them are not essentially different in principle from the solution which we have just given.

There is also a neat analytical solution by Mr. Wales in the Philosophical Transactions for 1781, where the angles BAm, CAm are calculated by means of the trigonometrical tables and the method of trial and error. Indeed those angles may be very readily found by this method, from Dr. Hutton's solution above; for we have only to divide the given angle BAC into two parts u(BAm) and v(CAm) so that

$$\frac{a \sin v}{r - a_,\cos v} = \frac{b \sin u}{r - b \cos u},$$ which may be easily done from the

tables, by a few trials. And in general this way of resolving questions of this kind is much better than by reducing the result to a final equation. L.

## Questions proposed in 1728, and answered in 1729.

I. QUESTION 130, by Mr. Rich. Whitehead.

Three persons owned a field, which they had used so long in common, as to have quite forgotten the bounds of their respective shares ; at length, after much searching, an old plan of the field was found ; but it was so much obliterated, that only the following particulars could be collected therefrom towards determining the three shares. The form was a trapezium ADEC, the side AD $=$ 6·4 chains, AE $=$ 14 chains, EC $=$ 10·1 chains. The three shares were triangular, formed by two right lines drawn from A and C to a point B in DE ; ∠ ABC was 78° ; the ratio of BC to BA as 3 to 4 ; and the ratio of BD to BE as 3 to 2. Required the sides of the triangles ?

Answered by Mr. Turner.

Having AC given $= 14$ chains, the angle ABC $= 78°$ and the ratio BA to BC as 4 to 3. It is easy to find BA $= 12\cdot51881$, and BC $= 9\cdot38912$. Draw the lines as in the scheme, and put

DA $= b = 6\cdot4$,
Ag $= g = 9\cdot4487$,
gc $= m = 4\cdot5512$,
gB $= h = 8\cdot2123$,
EC $= n = 10\cdot1$,
BC $= a$?
Ap $= e$?
EH $= y$?
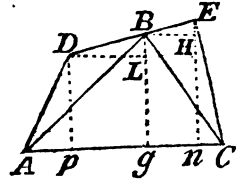
Then $2 : 3 :: m - a : g - e$ per question,
$4bb - 4ee = 4hh - 12hy + 9yy$ per Euclid 47. 1.
$9nn - ss - 4se - 4ee = 9hh + 18hy + 9yy$:
Putting $s = 3m - 2g$, and $w = 9nn - ss$,
We shall have $y = 2h - 2\sqrt{(ab - ee)}$,
and $y = \sqrt{\dfrac{w - 4se - 4ee}{9}} - h.$

And reducing the equation $e = 3\cdot43139$, y is found $= 1\cdot87239$, $a = 0\cdot53966$. Hence BD $= 6\cdot6411$, and BE $= 4\cdot4274$ chains, and the whole DE $= 11\cdot06850$.

	A.	R.	P.
ADB $=$	1	0	22·7667
ABC $=$	5	3	1·3731
CBE $=$	2	0	11·7145

And the areas of

Additional Solution.

Here being given of the trapezium the three sides AD, AC, and CE, the angle ABC formed in the other side, the ratio (m to n) of its legs AB, BC, and the ratio (p to q) of the segments BD, BE of the other side; from which the figure will be constructed thus:

Construction. On the given base AC (by Prob. 3 of Simpson's Algebra) constitute a triangle ABC whose sides shall be in the given ratio of m to n, and whose vertical angle shall be equal to the given one. Then with the centres A and c, and radii equal to the other two given sides of the trapezium, describe two circles. Produce CB, and make as $q : p :: $ CB : BF and also $:: $ CE : FD; which apply to the circumference of the circle whose centre is A. Draw DB, and continue it to meet the circumference of c in E; and ADEC is the trapezium required.
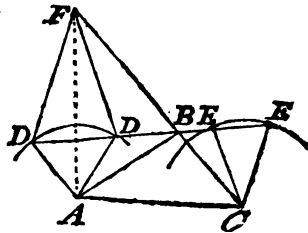
Demonstration. The triangles BCE, BFD are similar by Eucl. VI. 7, and therefore BE : BD :: BC : BF :: $q : p$ by the construction; and the rest is also evident from the construction.

Calculation. As $m + n : m - n$ :: cotang. $\frac{1}{2}$ $\angle$ ABC : tang. $\frac{1}{2}$ dif. of the angles BAC, BCA at the base; which added to and taken from half their sum, we obtain those angles themselves; and consequently the triangle ABC becomes all known, which is one part of the field. Then in the trapezium BADF, are known all the sides and the $\angle$ ABF; to find the diagonal BD; for AD is given, and AB and the $\angle$ B have been found; then by the construction $q : p$ :: $\begin{cases} \text{CB : BF} = \dfrac{p}{q} \times \text{CB} \\ \text{CE : DF} = \dfrac{p}{q} \times \text{CE} \end{cases}$

To determine BD, draw AF; then in the triangle ABF, given two sides AB, BF, and their included $\angle$, to find the third side AF and the $\angle$ AFB. Then in the triangle ADF, are given all the sides, to find the $\angle$ AFD; and the sum or difference of AFB and AFD will be the $\angle$ DFB. Wherefore in the triangle DFB, are known two sides DF, FB, and their included angle, and consequently the whole triangle; which is the 2d portion.——Lastly, as $m : n$ :: DB : BE; hence the 3d portion BBE is known.   H.

II.   QUESTION 131, *by Mr.* Tho. Dodd.

Suppose that in two geometrical series continued *ad infinitum*, the sum of all the terms of the first is to the sum of all the terms of second as 4 to 9; also, that the first two terms of the *first* series are 40 and 35, and that the second term of the *second* series is $46\frac{19}{36}$. Required the first term and ratio of the *second* series?

Answered by Mr. Tho. Grant.

The sum of the first geometrical progression, continued *ad infinitum*, is 320 (the first term being 40 and the second 35). Then by the question, as $4 : 9$ :: $320 : 720$ = the sum of the decreasing geometrical progression, whose second term is $46\frac{19}{36}$, and last term = 0.

Let x = the first term of the same; then, per Euclid, 5, 12, it will be as $x : 46\frac{19}{36}$ :: $720 : 720 - x$. This, turned into an equation, gives $720x - xx = 33500$; whence $x = 50$. and the common ratio as 6 to $5\frac{7}{12}$, or as 1 to $\frac{67}{72}$.

III.   QUESTION 131, *by Mr.* John Hartley.

Amongst the other ornaments of the famous temple of Solomon was the curious Molten Sea, which stood upon twelve brazen oxen: its form was cylindrical, and it contained 2000 baths; the outer

circumference of the cylinder was 30 cubits, and its internal depth 5 cubits, and the thickness of the metal was a hand breadth or palm. Now, supposing the Jewish cubit 21·888 inches, and the palm 3·648 inches; what was the measure of one bath in wine gallons? And, supposing the Molten Sea made of copper, what would be its weight; and what weight would press upon the oxen, when the cylinder was full of water?

Answered by Mr. Turner.

If a line of 30 cubits, or 656·64 inches, would surround the Molten Sea on the outside; then the outside diameter was but

                                         209·015 inches.

Twice the thickness of the copper or 2 hand br. sub............ ....   7·296

Remains the inside diameter ....   201·719
The depth 109·44 inches, area of the base   34311·982
The whole solid content from outside ... 3755103·31 cubic inches.
Add the copper's bottom..............   125170·11
The sum is ....   3880273·42
Content of the inside ....   3497523·126
Difference ....   382750·294

Equal to the cubic inches of copper. Divide the last sum but one by 2000, the number of baths it contained, gives the quotient 1748·76 cubic inches, the content of one bath $= 7\frac{7}{50}$ gallons wine measure; the whole contained 151400·8 gallons wine measure, or 60 tuns 20 gallons. Then according to Mr. Ward,

A cubic inch of $\begin{cases} \text{copper weighs} & 5·208369 \\ \text{clear water} & 0·578697 \end{cases}$ ounces avoirdupois.

Hence the weight of $\begin{cases} \text{water} \\ \text{copper} \\ \text{on the oxen} \end{cases}$

	tuns	hund.	pounds	
water	56	8	12	
copper	55	12	40	13
on the oxen	112	0	52	13

Answered by Mr. John Clark.

By Harris's Lexicon, a $\begin{cases} \text{cubit is} & 21·888 \text{ inches} \\ \text{palm} & 3·648 \\ \text{bath} & 1747·700 \end{cases}$

whence the quantity of 2000 baths   3495400
               water in the sea   3497504
                    difference   2104, which is but $\frac{6}{10}$ of an inch in the depth, or little more than one bath.

	T.	c.	st.	lb.	dw.	gr.
Then by Mr. Boyle's of water is..........	57	0	5	11	205	8
proportion the wt. sides of the copper ...	37	2	7	4	59	19
and the bottom in proportion to the sides	18	1	0	1	173	9
The whole weight in avoirdupois............	112	4	5	4	146	6

IV.   QUESTION 132, by Mr. Tho. Williams.

Supposing that xcm is the sector of a conic parabola, of which x is the focus, and c the principal vertex : there are given $xc = 26$ poles, and the area of the sector $= 676$ square poles.   Required the length and position of xm ?
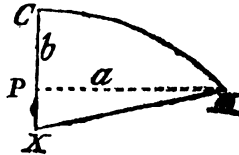
Answered by Mr. W. Milward.

Let fall the ordinate MP, which call a, and put $cx = b = 26$ poles.   As $4b : a :: a :$ $\frac{aa}{4b} = $ cP per conics. $\frac{2cP \times PM}{3} = $ the area of the semi-parabola cPM $= \frac{2aaa}{12b}$ or $\frac{aaa}{6b}$; $cx - cP = Px = \frac{4bb - aa}{4b}$, and $Px \times \frac{1}{2}MP$ $= $ the area of the triangle PXM.

The $\triangle$ PXM $+$ area of the semi-parabola $= $ the area of the sector, viz. $\frac{aaa}{6b} + \frac{4bba - aua}{8b} = bb$ per question.

$aaa + 12bba = 24bbb$; here a is $= 42.55$ poles. Hence $MX = 43.445$.

Additional Solution.

Putting $b = cx$, and $x = $ the ordinate PM; then, by the nature of the parabola, $cP = \frac{x^2}{4b}$; and, by prob. 7, page 321, Mensuration, $(cx + \frac{1}{3}cP) \times \frac{1}{2}PM = $ the area $= b^2$ by the question ; that is, $\left(b + \frac{x^2}{12b}\right) \times \frac{1}{2}x = b^2$, or $x^3 + 12b^2x = 24b^3$.   H.

V.   QUESTION 133, by Τελωνης Παλαι.

Find such a right-angled triangle that if double the ale area, be severally subtracted from each of its sides, the three remainders shall be rational squares.

Answered by Mr. Geo. Brown.

Let $a = $ side of the triangle ; then $\frac{2aa}{282} = $ the double area.   And suppose $a - \frac{2aa}{282} = 4$; which will admit of many answers, for

when brought to a solution it will be $a - 70\cdot5 = \sqrt{(4970\cdot25 - 564)}$, and the greatest root take to be the utmost extent mentioned, viz. $a = 705 + \sqrt{(4970\cdot25 - 564)} = 136\cdot879$. And making this equation for the other side $a - \dfrac{2aa}{282} = 2\cdot25$; then $a = 138\cdot71$.

And a vessel erected upon such a plane, at 10 inches deep, contains ·334·49 ale gallons.

Additional Solution.

It is not clear what is meant by this question as it is worded; if double the area of the triangle is to be taken from each side, and the three remainders are to be squares, then the solution may be as follows:

Let the two legs of the triangle be denoted by $\dfrac{x}{x+y}$ and $\dfrac{y}{x+y}$, then $\dfrac{xy}{(x+y)^2}$ will express the double area, and this being subtracted from each of the legs leaves $\dfrac{x}{x+y} - \dfrac{xy}{(x+y)^2} = \dfrac{x^2}{(x+y)^2}$ and $\dfrac{y}{x+y} - \dfrac{xy}{(x+y)^2} = \dfrac{y^2}{(x+y)^2}$, which are both rational squares. It therefore only remains to make the difference between the double area and the hypothenuse a rational square. The hypothenuse is obviously expressed by $\dfrac{\sqrt{(x^2+y^2)}}{x+y}$, and therefore by the question $\dfrac{\sqrt{(x^2+y^2)}}{x+y} - \dfrac{xy}{(x+y)^2} = \dfrac{(x+y)\sqrt{(x^2+y^2)} - xy}{(x+y)^2}$ is to be a square, or $(x+y)\sqrt{(x^2+y^2)} - xy$ must be a square. Take $x = m^2 - n^2$ and $y = 2mn$, then $(x+y)\sqrt{(x^2+y^2)} - xy = (m^2 - n^2 + 2mn)(m^2 + n^2) - 2mn(m^2 - n^2) = m^4 + 4mn^3 - n^4 = $ a square. Put $m^4 + 4mn^3 - n^4 = m^4$, whence $4m = n$, or $\dfrac{m}{n} = \frac{1}{4}$, therefore we may take $m = 1$ and $n = 4$. But the preceding values of m and n would give x negative; therefore take $m = 1 + v$ and $n = 4$, and then $m^4 + 4mn^3 - n^4 = v^4 + 4v^3 + 6v^2 + 260 v + 1 = $ a square. Assume $v^2 - 130v + 1$ for the root, that is put $v^4 + 4v^3 + 6v^2 + 260v + 1 = (v^2 - 130v + 1)^2 = v^4 - 260v^3 + 16898v^2 + 260v + 1$, whence $v = \frac{16992}{264} = \frac{4223}{66}$; $m = 1 + v = \frac{4289}{66}$ and $n = 4 = \frac{264}{66}$, and therefore we may take $m = 4289$ and $n = 264$; whence $x = m^2 - n^2 = 18325825$ and $y = 2mn = 2264592$: and hence the legs of the triangle will be found to be $\dfrac{x}{x+y} = \frac{18325825}{20590417}$, and $\dfrac{y}{x+y} = \frac{2264592}{20590417}$.

It is very obvious that the expression $m^4 + 4mn^3 - n^4$ will be a square when $m = n$, but this relation of m and n would give $x = m^2 - n^2 = 0$; therefore take $m = n + z$ and then $m^4 + 4mn^3 - n^4 = 4n^4 + 8n^3z + 6n^2z^2 + 4nz^3 + z^4 = $ a square $= \left(2n^2 + 2nz + \dfrac{z^2}{2}\right)^2 = 4n^4 + 8n^3z + 6n^2z^2 + 2nz^3 + \dfrac{z^4}{4}$; whence $z = -\dfrac{8n}{3}$

$m = n + z = -\dfrac{5n}{3}$, or $\dfrac{m}{n} = \dfrac{-5}{3}$: hence it appears that $m = 5$, and $n = -3$ will make the expression $m^4 + 4mn^3 - n^4$ a square. But the preceding values of m and n would manifestly give y negative. In order therefore to obtain positive values for x and y take $m = v + 5$ and $n = v - 3$, then $m^4 + 4mn^3 - n^4 = 4v^4 + 112v^3 + 26v^2 + 944v + 4 = $ a square; take $2v^2 - 236v - 2$ for its root, then $4v^4 + 112v^3 + 26v^2 + 944v + 4 = (2v^2 - 236v - 2)^2 = 4v^4 - 944v^3 + 45688v^2 + 944v + 4$; whence $v = \dfrac{45714}{1056} = \dfrac{7619}{176}$, $m = v + 5$

$= \dfrac{8499}{176}$, $n = v - 3 = \dfrac{7091}{176}$, whence $x = m^2 - n^2 = 21950720$ and $y = 2mn = 120532818$: and hence the legs of the triangle are $\dfrac{x}{x + y} = \dfrac{21950720}{142483538}$, and $\dfrac{y}{x + y} = \dfrac{120532818}{142483538}$.

Other solutions are given at question 638. L.

VI. QUESTION 134, by Mr. Tho. Grant.

Required the greatest space or area which can be included by four right lines, whose lengths are 20, 16, 12, and 10?

Answered by Mr. Tho. Grant.

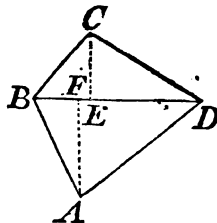Let $x = $ BD, $a = $ AD, $b = $ AB, $c = $ CD, and $d = $ BC.
　　　　1. By the figure,

$x : a + d :: a - d : \dfrac{aa - dd}{x} = $ DF $-$ BF,

therefore DF $= \dfrac{xx + aa - dd}{2x}$.

　　　2. Again by the figure.

$x : c + b :: c - b : \dfrac{cc - bb}{x} = $ DE $-$ BE,

therefore DE $= \dfrac{xx + cc - bb}{2x}$.

3. By Eucl. I. 47.

$$\sqrt{\dfrac{2a^2x^2 + 2d^2x^2 + 2a^2d^2 - x^4 - a^4 - d^4}{4xx}} = \text{FA.}$$

4. By the same,

$$\sqrt{\dfrac{2c^2x^2 + 2b^2x^2 + 2c^2b^2 - x^4 - c^4 - b^4}{4xx}} = \text{EC.}$$

5. By the figure,

$$\tfrac{1}{4}\sqrt{(2a^2x^2 + 2d^2x^2 + 2a^2d^2 - x^4 - a^4 - d^4)} +$$
$$\tfrac{1}{4}\sqrt{(2c^2x^2 + 2b^2x^2 + 2c^2b^2 - x^4 - c^4 - b^4)} = \text{the greatest area.}$$

6. The last in fluxions,

$$\dfrac{a^2x\dot{x} + d^2x\dot{x} - x^3\dot{x}}{2\sqrt{(2a^2x^4 + 2d^2x^2 + 2a^2d^2 - x^4 - a^4 - d^4)}} +$$
$$\dfrac{c^2x\dot{x} + b^2x\dot{x} - x^3\dot{x}}{2\sqrt{(2c^2x^4 + 2b^2x^2 + 2c^2b^2 - x^4 - c^4 - b^4)}} = 0.$$

7. This reduced is

$$\left.
\begin{array}{l}
-x^8 \begin{array}{l} +2a^2 \\ +2b^2 \\ +2c^2 \\ +2d^2 \end{array}
x^6 \begin{array}{l} -a^4 \\ -b^4 \\ -c^4 \\ -d^4 \\ -4a^2b^2 \\ -4a^2c^2 \\ -4b^2d^2 \\ -4c^2d^2 \end{array}
x^4 \begin{array}{l} +2a^2b^4 \\ +2a^2c^4 \\ +2a^4b^2 \\ +2a^4c^2 \\ +2b^2d^4 \\ +2b^4d^2 \\ +2c^2d^4 \\ +2c^4d^2 \end{array}
x^2 \begin{array}{l} -b^4a^4 \\ -b^4d^4 \\ -c^4d^4 \\ -c^4a^4 \\ +4a^2b^2c^2d^2 \end{array}
\end{array}
\right\} = 0.$$

8. Or in numbers

$$-x^8 + 1800x^6 - 1076272x^4 + 222272000x^2 - 8768000000 = 0.$$

Hence $x = 21\cdot187$. And the greatest area is $194\cdot4$, &c.

Additional Solution.

By theorem 12 of Simpson on the Max. and Min. of Geometrical Quantities, the trapezium will be greatest when it can be inscribed in a circle; and then, by rule 5, page 72, of my Mensuration, the area will be found thus: Half the sum of the given sides is 29, from which each of the sides being taken, we have the four remainders 9, 13, 17, 19; and the area $= \sqrt{9 \times 13 \times 17 \times 19} = 194\cdot4$.

Another Additional Solution.

It appears from the additional solutions to quest. 56, that the area will be the greatest possible when the trapezium is inscribed in a circle; and the same result may be obtained by a fluxionary process as follows:

Retaining the same substitutions as in the first of those solutions, we have

$ad \sin \varphi + bc \sin \theta = e \ldots (1). \quad ad \cos \varphi - bc \cos \theta = r \ldots (2).$

Taking fluxions, we get from the first equation $\dfrac{\dot{\varphi}}{\dot{\theta}} = -\dfrac{bc \cos \theta}{ad \cos \varphi}$,

and from the second $\dfrac{\dot{\varphi}}{\dot{\theta}} = \dfrac{bc \sin \theta}{ad \sin \varphi}$.

Hence $\dfrac{-\cos \theta}{\cos \varphi} = \dfrac{\sin \theta}{\sin \varphi}$, or $\dfrac{-\cos \theta}{\sin \theta} = \dfrac{\cos \varphi}{\sin \varphi}$.

Consequently $\theta + \varphi = 180°$, for then $-\cos \theta = \cos \varphi$, and $\sin \theta = \sin \varphi$; and therefore the figure is inscribed in a circle.

Also the above equations then become

$$(ad + bc) \sin \varphi = e, \qquad (ad + bc) \cos \varphi = r;$$
$$\text{therefore } (ad + bc)^2 (\sin^2\varphi + \cos^2\varphi) = e^2 + r^2,$$

and $e^2 = (ad + bc)^2 - r^2$

$\quad = \frac{1}{4}(a+d+c-b)(a+d+b-c)(b+c+a-d)(b+c+d-a);$

therefore, putting $s = $ half the sum of the four sides, the maximum area is

$$= \sqrt{\Big\{(s - a)(s - b)(s - c)(s - d)\Big\}}. \quad \text{J. L.}$$

VII. QUESTION 135, *by Mr.* John Turner.

There is a very lofty mountain whose height is such, that when cubed and multiplied into three times the square root of the length of a tangent from its vertex to the earth's surface, the product will be 1005·9309 miles. What is the mountain's height, the semi-diameter of the earth being 3963 miles?

Answered by Mr. John Bulman.

Let $x = $ BA $= $ the mountain's height.
Then $x + 3963 = $ AC.   *See Fig. to Quest.* 101:
Hence $xx + 7926x + 15705369 = $ AC^2.   *page* 130.
Or $xx + 7926x = $ AM^2.
Consequently $\sqrt[3]{(81x^{14} + 642006x^{13})} = 1005 \cdot 9309.$

Then if the latter part of the equation at the last step be involved to the fourth power, and the radical sign taken away from the first part, the equation will be clear of surds, and x may be found $= 3$ miles, the mountain's height required.

The same answered by Mr. R. Fearnside.

Put $a = $ AB, $b = $ CB $= 3963$, $e = $ AM, $c = 1005 \cdot 9309$. Then

$3a^2\sqrt{e} = c$, and $e = \dfrac{cc}{9a^v}$. Then $ee = 2ba + aa = \dfrac{c^4}{81a^{12}}$, or

$a^{14} + 2ba^{13} = \dfrac{c^4}{81}$; reduced, $a = 2{\cdot}999999$ miles.

<h3 style="text-align:center">THE PRIZE QUESTION.</h3>

There is a spheroidal cask which contains 160 barrels of beer; the head diameter is equal to the latus rectum of the generating ellipse; and the length exceeds the diagonal taken at the bung by 26 inches. Required its dimensions?

Answered by Mr. Geo. Brown.

Let AC $= a$, CE $=$ LD $= x$, EE $-$ AD $= b = 26$ inches, $d = 5760$ the content, $c = 1077{\cdot}15$. $4xx - 4bx + bb = $ AD^2.
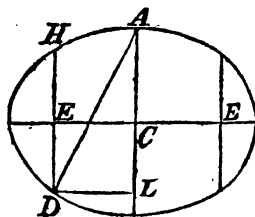
$\sqrt{(3xx - 4bx + bb)} - a = $ DE.

$3xx - 4bx + bb + aa - 2a\sqrt{(3xx - 4bx + bb)} = $ DE^2.

$12xx - 16bx + 4bb + 4aa - 8a\sqrt{(3xx - 4bx + bb)} = $ DH^2.

$$\dfrac{24x^3 - 32bx^2 + 8bbx + 24aax - 16ax\sqrt{(3xx - 4bx + bb)}}{c} = d.$$

$24x^3 - 32bx^2 + 8bbx + 24aax - dc = 16ax\sqrt{(3xx - 4bx + bb)}$, and when squared $576x^6 - 1536bx^5 + 1408b^2x^4 + 1152a^2x^4 - 512b^3x^3 - 48dcx^3 - 1536ba^2x^3 + 64b^4x^2 + 64bdcx^2 + 384b^2a^2x^2 + 576a^4x^2 - 16dcx - 48dca^2x + d^2c^2 = 768a^2x^4 - 1024ba^2x^3 + 256b^2a^2x^2$.

And when aa is found in the unknown quantity x, there will be an equation of the 12th power affected; from whence the length is 155$\cdot$2, diagonal 130$\cdot$2, bung diameter 128$\cdot$3, and head 82$\cdot$1. *Q. E. I.*

Others have answered it as below.

	Length	Diag.	Bung	Head
Mr. *John Bulman*...............	157·25	131·25	128·8	81·72
Mr. *James Clarke*...............	160	134	127	88
Mr. *John Rosser*...............			126·01	76·5
Mr. *John Finch*...............	157		128·19	81·5
Mr. *John Side-Bottom*..........	163	137		81·5
Mr. *John Hampson*............	157		128	·81

Questions proposed in 1729, and answered in 1730.

I.   QUESTION 136, by Mr. John Bulman.

The distance from London to York is 150 miles, and Stilton divides the distance in extreme and mean proportion, the greater part being the distance from Stilton to York : Now suppose two messengers, dispatched at the same time from London, to travel uniformly to Stilton and York, how must their rates of travelling be adjusted so as to arrive at those two places at the same time ?

Answered by Mr. Rob. Fearnside.

The distance between York and London given 150 miles; from London to Stilton will be easily found 57·2949 miles; and Stilton to York 92·7051.   Put $y =$ the space passed over by the courier bound to Stilton, in any moment of time ; and $x =$ the space passed over by him bound for York in the same moment of time.   Then, per question, $xx - 2xy + yy = xy$, in fluxions $2x\dot{x} - 3x\dot{y} - 3y\dot{x} + 2y\dot{y}$ $= 0$.   Reduced to an analogy, $\dot{y} : \dot{x}$ (the fluxion of any flowing quantity being taken for its velocity) $:: 2x - 3y : 3x - 2y$.   That is, as 128·1242 to 335·4161.

Additional Solution.

The meaning of this question is, To divide 150 miles into two parts in extreme and mean proportion ; and to find the ratio of the equable velocities, by which the less part and the whole length may be passed over in the same time.

Put $x =$ the less part, and $a = 150$ the whole.   Then $a - x =$ the greater part; and, by the question, $x : a - x :: a - x : a$;

hence $ax = a^2 - 2ax + x^2$, and $x = \dfrac{3 - \sqrt{5}}{2} \times a =$ the less part.

Therefore $a - x = \dfrac{\sqrt{5} - 1}{2} \times a =$ the greater part.——Hence

The two parts of any quantity which is divided into extreme and mean proportion, are in the ratio of $\dfrac{3 - \sqrt{5}}{2}$ to $\dfrac{\sqrt{5} - 1}{2}$, or of

$3 - \sqrt{5}$ to $\sqrt{5} - 1$, or of 1 to $\dfrac{\sqrt{5} + 2}{2}$, or of $\dfrac{\sqrt{5} - 2}{2}$ to 1.

And the less part to the whole, as $\dfrac{3 - \sqrt{5}}{2}$ to 1 ; which is the ratio of their velocities or rates of travelling required.   H.

M 2

II.   QUESTION 137, *by Mr.* John Simmons.

Suppose the side or edge of each of the five regular solids to be 30 inches, what will be the content of each in wine gallons; also what will be the content of each of their respective inscribed and circumscribing spheres ?

Answered by Mr. Tho. Grant.

Names of the bodies.	Capacities	Inscribed spheres	Holds besides the spheres	Ambient spheres content
	Wine gall.	Wine gall.	Wine gall.	Wine gall.
Tetrahedron	13·77	4·16	9·61	112·43
Hexahedron	116·88	61·20	55·68	318·00
Octohedron	55·09	33·08	22·01	173·09
Dodecahedron	895·69	676·00	219·69	1347·09
Icosahedron	255·00	211·34	43·65	421·17

Remark. In the *Scholia* at page 403, &c. of my Mensuration, the true values of the content of each regular solid, with that of the radius of the inscribed and of the circumscribed sphere, are expressed in terms of the linear edge or side; whence the numbers, as above, may be easily found.   H.

III.   QUESTION 138, *by Mr.* Tho. Grant.

A vessel under sail goes 25 Grecian feet in $1\frac{1}{2}$ seconds, and in the same time it was observed that a ball fell from the mast head upon the deck. What was the space through which the ball fell ?

Answered.

The ball descending with a compound force, that is, an horizontal force by the motion of the ship, and by the force of gravity; the spaces described by such motions are analogous to the abscissas and their respective semi-ordinates, in the conic parabola, and the motion will be in the curve of the same.

Then there is given the abscissa $= x = 434\cdot25$ inches $=$ the space described in three half seconds by the force of gravity only; and its semi-ordinate $= y = 25$ Grecian feet $= 302\cdot1$ inches English. Whence by fluxions

$x + \dfrac{5y^2}{8x} + \dfrac{3y^4}{128x^3} - \dfrac{3y^6}{256x^5} + \dfrac{y^8}{2048x^7}$ &c. $=$ 567·42 inch. or 47·285 feet, the space the ball descended.

Additional Solution.

The 25 Grecian feet are $=$ 25 × 1·007 or 25·175 English feet $=$ the horizontal motion, or the ordinate of the parabola, which call $2y$; and as $1^2 : (\tfrac{3}{2})^2$, or as 4 : 9 :: $16\tfrac{1}{12}$ feet : $36\tfrac{3}{16}$ feet $=$ the perpendicular descent, or the abscissa of the parabola, which call x. Then, by Cor. 1, page 310, of my Mensuration, the true length of the curve will be $\sqrt{(x^2 + y^2)} + \dfrac{y^2}{x} \times$ hyp. log. of $\dfrac{x + \sqrt{(x^2 + y^2)}}{y} =$ 45·9578 feet $=$ the length of the line described by the ball. ·H.

IV. QUESTION 139, by Mr. Tho. Williams.

In the triangle ACB, the side AB is 21 chains, and the $\angle$BAC $=$ 82°; a right line AD divides the angle BAC into the angles BAD, DAC having the ratio of 2 to 3, and AD divides BC so that BD : DC :: 3 : 2. Required the sides AC and BC, together with the areas of the triangles BAD and DAC?

Answered by Mr. Rob. Fearnside.

Put AB $=$ 21 chains $= b$, the natural sine of the angle BAC (the radius being an unit) $=$ 82° $=$ 0·990268 $= t$, the natural sine $\angle$CAD $=$ 49° 12′ $=$ ·756995 $= s$; and BC $= e$.

Then as $e : t :: b : bt \div e =$ sine $\angle$ACD. Again, as $s :$ CD $(= \tfrac{2}{5}e) :: bt \div e :$ AD$=2bt \div 5s=10\cdot988$. On AC let fall the perpendicular Bp. And say as $b : 1 :: t :$ Bp $=20\cdot795$. Consequently Dd is $=8\cdot318=\tfrac{2}{5}$Bp, CD$=13\cdot1837$, the area of the $\triangle$ ACD$=4\cdot166$ acres, and of the $\triangle$ ADB is $= 6\cdot249$ acres.
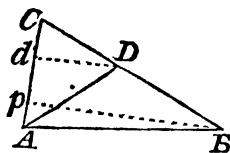
Additional Solution.

The $\angle$BAD $= \tfrac{2}{5}$ of 82° $=$ 32° 48′, whose sine call a; and $\angle$CAD $= \tfrac{3}{5}$ of 82° $=$ 49° 12′, whose sine call s; also $t = s$. $\angle$BAC or 82°, and $b =$ AB $=$ 21 chains as above. Then, also as above, AD $= 2bt \div 5s =$ 10·9885: And hence $\tfrac{1}{2}$AB × AD × s. $\angle$BAD or 32° 48′ $= abbt \div 5s =$ 62·502 square chains $=$ 6·2502 acres $=$ $\triangle$ ABD.

Again, as AB $+$ AD : AB $-$ AD :: cot. $\tfrac{1}{2}$ $\angle$BAD or cot. 16° 24′

: tang. $\dfrac{\angle \text{ADB} - \angle \text{B}}{2} = 45° \; 5'$; which added to $73° \; 36'$ half the

sum of these two angles, we have $118° \; 41' =$ the $\angle$ ADB ; and this
taken from $180°$, there remains $61° \; 19'$ for the $\angle$ ADC. Then, as s.
$\angle$ BDA : s. $\angle$ BAD :: AB : BD $= 12\cdot967$; the $\frac{2}{3}$ds of which is DC $=$
$8\cdot645$. Hence $\frac{1}{2}$AD $\times$ DC $\times$ s. $\angle$ ADC $= 41\cdot648$ chains $= 4\cdot1648$
acres for the other part ACD. And, lastly, as s. $\angle$ CAD : s. $\angle$ D :: CD
: CA $= 10\cdot0135$.

Construction.

Having drawn the given side AB, and the lines AD, AC making the
given angles BAD, DAC; in BA pro-
duced take AE to AB in the given
ratio of CD to DB, that is, of 2 to 3;
draw EC parallel to AD; lastly draw
CB, and it is done.
    For, by sim. $\triangle$s, AB : AE :: BD
: DC.   H.
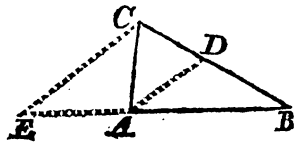
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v. QUESTION 140, *by Mr.* John Lowe.

   Suppose there be a pyramid erected upon a triangular base, whose
longest side is eighty-eight inches, and the angle opposite to the
longest side is eighty-five degrees and one minute, and the rectangle
of the other two sides added to their sum is four thousand one hun-
dred and three inches ; and the pyramid's greatest angle of altitude,
is in proportion to its least, as seven is to six: What is the difference,
in alc gallons, betwixt the greatest cylinder and sphere that may be
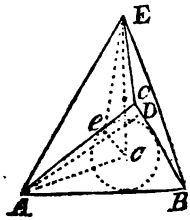inscribed in the said pyramid ?

Answered by Mr. Lowe, *the proposer.*

   Put AB $= b$, the sine of the angle ACB $= s$, its cosine $= c$, 4103
$= p$, AC $= a$, and BC $= e$; the radius $= 1$.
As $1 : a :: s : sa =$ AD; and $1 : a :: c :$
$ca =$ CD: but $ae + a + e = p$; hence $e =$
$\dfrac{p - a}{a + 1}$, and $\dfrac{p - a}{a + 1} - ca = $ BD $= \dfrac{p - a - ca^2 - ca}{a + 1}$:

Put $1 + c = x$, then $\dfrac{p - xa - ca^2}{a + 1} = $ BD.

Hence $\dfrac{c^2a^4 + 2xca^3 - 2pca^2 + xxaa - 2pxa + pp}{aa + 2a + 1}$

$+ \; ssaa = bb$. Which equation reduced gives $a = 75$; and thence
$e = 53$.

For the pyramid's altitude. The radius of the inscribed circle is 18·3315 inches $= r$. And $AC = 57·97468$ inches $= b$; $a =$ pyramid's height $= EC$. Then $\sqrt{(aa + rr)} = E\theta$ the lesser hypothenuse, and $\sqrt{(aa + rr)} : 1 :: a : \dfrac{a}{\sqrt{(aa + rr)}} =$ the sine of the greater angle of altitude. Then $7 : 6 :: \dfrac{a}{\sqrt{(aa + rr)}} : \dfrac{6a}{7\sqrt{(aa + rr)}} =$ the sine of the least angle of altitude; also $\dfrac{6a}{7\sqrt{(aa + rr)}} : a :: 1 : \frac{7}{6}\sqrt{(aa + rr)} = AE$ the greater hypothenuse. And $aa + bb = \dfrac{49aa + 49rr}{36}$; which gives $a = 89·6712$ the pyramid's height.

The height of the greatest cylinder is found $= 29·8904$, being $\frac{1}{3}$d of the pyramid's height, the content 49·733 ale gallons; the diameter of the sphere 30·5012, the content 52·687 ale gallons; the difference 2·953 ale gallons.

VI.   QUESTION 141, *by* Mr. Will. Browne.

Find three numbers such that the square of the first added to the product of the first and second may be 20; the square of the second added to the product of the second and third may be 30; and the square of the third added to the product of the first and third may be 40.

Answered by Mr. Lovatt.

This question may be found in Mr. Kersey's Algebra, p. 299. We have given $aa + aa = 20$, $ee + ey = 30$, and $yy + ya = 40$; which will give $\sqrt{(40 + \frac{1}{4}aa)} - \frac{1}{2}a = \dfrac{70aa - 400 - a^4}{20a - aaa}$; and reduced is $a = 3·051$, $e = 3·526$, $y = 4·98$.

Additional Solution.

Put $b = 20$, $c = 30$, and $d = 40$, then
$$a^2 + ae = b, \qquad e^2 + ey = c, \qquad y^2 + ay = d,$$
or putting $e = ma$ and $y = na$
$$a^2(1 + m) = b, \text{ or } a^2 = \frac{b}{1 + m},$$
$$a^2(m^2 + mn) = c, \text{ or } a^2 = \frac{c}{m^2 + mn},$$
$$a^2(n^2 + n) = d, \text{ or } a^2 = \frac{d}{n^2 + n}:$$

M 4

whence $\dfrac{b}{1+m} = \dfrac{d}{n^2+n}$ and $\dfrac{c}{m^2+mn} = \dfrac{d}{n^2+n}$,

$$\text{or } b(n^2 + n) = d(1 + m)$$
$$\text{and } c(n^2 + n) = d(m^2 + mn).$$

From the first of these equations $m = \dfrac{b(n^2 + n) - d}{d}$, which substituted for m in the second equation gives

$$c(n^2+n) = (bn^2 + bn - d)\left(\frac{bn^2 + bn - d + dn}{d}\right).$$

And by reduction

$$bn^4 + (db+2b^2)n^3 + (b^2-db-dc)n^2 - (d^2+2bd+cd)n + d^2 = 0,$$

or in numbers and dividing by 400, '

$$n^4 + 4n^3 - 4n^2 - 11n + 4 = 0.$$

From whence n is found $= 1.6369136$; then $m = \dfrac{b(n^2+n)-d}{d}$

$= 1.1581999$; $a = \sqrt{\dfrac{b}{1+m}} = 3.0441718$; $c = ma = 3.5257595$;

and $y = na = 4.983462$.   L.

VII.   QUESTION 142, *by Mr.* John Ingleborough.

The great tun at Heidelberg, an author says, held as much wine in
its entrails as the Colossus of Rhodes held water between its thighs.
It is 31 feet long and 21 high.

Now supposing the tun to be of the same shape with Parson's
cask, mentioned last year, so far as may agree with the dimen-
sions above; how many gallons of wine would it contain? Se-
condly, supposing Parson's cask of stout to equiponderate with the
Heidelberg tun of wine, suspending from a balance 60 feet long;
what would the difference of the brachia of the balance be? And
thirdly, supposing it were required that each vessel should evacuate
itself exactly in two hours by a hole in the lowest place; what must
the diameter of each bore be, that so the vessels may hang in equilibrio
all the time the liquor is running out, not admitting the weight of the
vessels to interfere or make any variation?

Answered by Mr. Rich. Lovatt.

The length 372 inches $= d$, bung diameter $= 252 = b$. If the
cask be the same shape as Parson's, the head is the latus rectum; then
$\sqrt{(dd-bb)} : b :: b : 141.334 =$ the head per conics: finding an in-
finite number of ordinates betwixt head and bung, will constitute the
solidity $= 59780.4$ wine gallons, and a mean of all those ordinates
will give an equal cylinder $= 217.3072$ inches. A gallon of wine

weighs 8·29645 pounds avoirdupois. Then 8·296 × 59780·4 = 495965·1593604, avoird. the whole weight. A gallon of stout weighs 10·25 pounds avoird. × 5760 = 59040, the weight of the liquor in Parson's cask. Then as the sum of the weights is to the difference, so is the length of the balance to the difference of brachia = 47·2348 feet. From Parsons's cask the liquor flows 128·8 inches, equal bung diameter, in 34·218 thirds of the time. Heidelberg cask flows 252 inches in 47·863 thirds. Then put $d = 432000$, the thirds in 2 hours, $n = 34·218$, $a =$ the drain of the bore in Parsons's cask. And then $\dfrac{128·8daa}{359·05n} = 5760$; hence $a = 1·127$ inch. And by the same way the bore is found 2·73119 inches in Heidelberg cask. Mr. *Tho. Grant* makes the bores 3·387 and 1·3518 inches.

Remark. The dimensions of the apertures, as determined in the solution to this question, may perhaps be pretty near the truth: but as for the mathematical certainty, there does not seem to be a possibility of it in the case.   H.
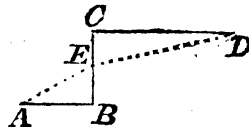
<center>THE PRIZE QUESTION.</center>

A courier at Montpellier receives orders to proceed with all possible expedition to Turin: Now Montpellier is situated 40 miles directly west of the Rhone, and Turin is 150 miles east of that river, whose course is directly south. The shortest distance between the two cities is 200 miles, but this is not the most expeditious route; for on the western side of the river the courier can travel at the rate of three miles an hour; but on the other side, on account of the difficulty of the roads he can only travel at the rate of two miles an hour. It is required to determine to what point of the river the courier must direct his course, so as to perform the journey in the least time possible?

<center>*Answered by Mr.* John Fearnside.</center>

Put AB $= c$, CD $= d$, BC $= b$, and CE $= b - x$. Then, per 47 Euc. I.   AE $= \sqrt{(cc + xx)}$; and DE $=$ $\sqrt{(dd + bb - 2bx + xx)}$. Then, by the question, $2 : 1 :: \sqrt{(dd+bb-2bx+xx)}$ $: \sqrt{\dfrac{dd + bb - 2bx + xx}{4}}$, and $3 : 1 ::$ $\sqrt{(cc + xx)} : \sqrt{\dfrac{cc + xx}{9}}$. Therefore $\sqrt{\dfrac{dd + bb - 2bx + xx}{4}} + \sqrt{\dfrac{cc + xx}{9}} = 0$: which thrown into

fluxions, and reduced, $x = 18\cdot5723$, $\text{CE} = 43\cdot8776$, $\text{AE} = 44\cdot1015$, and $\text{DE} = 156\cdot286$; and the time in going the whole journey 92^{h} $50^{\text{m}} 30^{\text{s}}$.

Remark. This problem may be seen applied to various and very curious purposes in *Hayes's* Fluxions.   H.

Questions proposed in 1730, and answered in 1731.

I. QUESTION 143, by Mr. Tho. Battersbye.

There is a right-lined triangle, in which a circle is inscribed; and in the corners of the triangle are placed three little circles, so drawn as to touch the adjacent sides of the triangle, and the greater circle, whose diameters are 484, 441, and 400. It is required to find the sides of the triangle, and the diameter of the greater circle?

Answered by Mr. Rich. Lovet.

Given $\begin{cases} \text{NH} = 220\cdot5 = dd, \\ \text{GC} = 242 \quad = nn, \\ \text{FM} = 200 \quad = hh, \\ \text{EL} = aa. \end{cases}$

Per 47 Eucl. I. we have $a^4 + 2aadd + d^4 - a^4 + 2aadd - d^4 = 4aadd = \text{HD}^2$, and $\therefore 2ad = \text{HD}$.

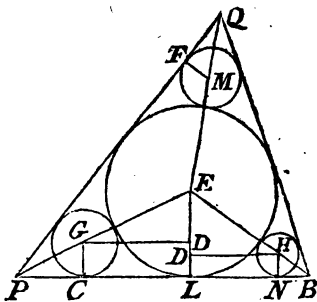Again $aa - dd : 2ad :: aa : \dfrac{2a^3d}{aa - dd} = \text{BL}$.

Hence $\dfrac{2a^3d}{aa - dd} + \dfrac{2a^3n}{aa - nn} =$

BP, $\dfrac{2a^3d}{aa - dd} + \dfrac{2a^3h}{aa - hh} =$ BQ, $\dfrac{2a^3n}{aa - nn} + \dfrac{2a^3h}{aa - hh} =$ PQ.

Or BP $= \dfrac{2a^3d - 2a^3dn^2 + 2a^5n - 2a^3nd^2}{(aa - dd) \times (aa - nn)}$;

Suppose $m = 2d + 2n$, $k = 2n^2d + 2nd^2$:

And PQ $= \dfrac{2a^5h - 2a^3hn^2 + 2a^5n - 2d^3h^2n}{(aa - nn) \times (aa - hh)}$;

Suppose $c = 2h + 2n$, $p = 2hn^2 + 2nh^2$.

Hence $\dfrac{\text{BP} + \text{PQ} + \text{QB} - 2\text{PB}}{2} \times \dfrac{\text{BP} + \text{PQ} + \text{QB} - 2\text{PQ}}{2} =$

$$\dfrac{4a^2dh}{(aa - dd) \times (aa - hh)}.$$

But $aa + nn : 1 :: aa - nn : \dfrac{aa - nn}{aa + nn} = $ s. $\angle$EGD; and, by that curious theorem 31, in Spherical Triangles explained by Ozanam, vol. II. p. 142, which holds good in all triangles, we have

$\dfrac{a^4cm - aack - aamp + pk}{a^4 - 2aann + n^4} : 1 :: 4dh : \left(\dfrac{aa - nn}{aa + nn}\right)^2$; Hence

$a^4cm - 4a^2dh - aack - aamp - 8aanndh = 4n^4dh - kp$. Put $b = ck + mp + 8nndh$, and $l = cm - 4dh$;

Then $aa = \dfrac{b}{2l} + \sqrt{\left(\dfrac{4n^4dh - kp}{l} + \dfrac{bb}{4ll}\right)} = 661 = $ EL.

And the lines are $2aa = 1322$, PB $= 2407\cdot85$, BQ $= 2188\cdot48$, PQ $= 2304\cdot72$.

Additional Solution.

Instead of supposing the figure a triangle, we may, without increasing the difficulty, suppose it to have any number of sides whatever. The problem thus generalized may be expressed thus.

A figure of any number of sides being such as to admit of having a circle inscribed in it: let there be given the radii of circles inscribed in each of its corners, so as to touch the adjacent sides of the figure and the inscribed circle: it is thence required to determine the sides of the figure and the radius of the inscribed circle?

Suppose perpendiculars to be drawn from the centre of the inscribed circle to all the sides of the figure; these will be radii of that circle, and the angle contained by every two adjoining perpendiculars will be the supplement of the opposite angle of the figure.

Let the fourths of these supplements be denoted by the letters

$$\alpha, \ \alpha', \ \alpha'', \ \alpha''', \ \&c.$$

the tangents of these angles by

$$t, \ t', \ t'', \ t''', \ \&c.$$

Let the radii of the circles inscribed in the corners of the figure be

$$r, \ r', \ r'', \ r''', \ \&c.$$

And let x be the radius of the circle which touches them all and also the sides of the figure.

Then, by a little consideration, it will readily appear that $\text{rad}^2 :$ $\tan^2\alpha :: x : r$; hence we have the following series of equations;

$$\left.\begin{array}{l} xt^2 = r \\ xt'^2 = r' \\ xt''^2 = r'' \\ \qquad \&c. \end{array}\right\} \quad \dots\dots\dots\dots (1)$$

the number of which is equal to the number of sides of the figure.
Moreover, because

$$\textit{a} + \textit{a}' + \textit{a}'' + \textit{a}''' + \&c. = \text{a right angle,}$$

if B be put for the sum of the products of the tangents t, t', t'', t''',
&c. taken two by two, (that is, let $B = tt' + tt'' + \&c. + t't'' + \&c.$), D
for the sum of their products taken four by four ; F for the sum of
their products taken six by six, &c. then, by a theorem in the arith-
metic of sines which is demonstrated in the solution to question 388.
(Also in the Mathematical Repository, quest. 273, vol. 3.).

$$1 - B + D - F + \&c. = 0 \dots\dots\dots\dots\dots(2).$$

By taking the products of the equations (1) two by two, four by
four, &c. we get

$$\begin{array}{ll} xtt' = \sqrt{(rr')}, & xt't'' = \sqrt{(r'r'')} \ \&c. \\ xtt'' = \sqrt{(rr'')}, & xt't''' = \sqrt{(r'r''')} \ \&c. \\ \qquad \&c. & \qquad \&c. \end{array}$$

$$x^2 t't''t''' = rr'r''r''' \ \&c.$$
$$\&c.$$

Let b be now put for the sum of the square roots of the products
of the known quantities r, r', r'', &c. taken two by two (that is
let $b = \sqrt{(rr')} + \sqrt{(rr'')} + \&c. + \sqrt{(r'r'')} + \&c.$) also let d be
put for the sum of the square roots of the products taken four by
four, and f for the sum of the square roots of their products taken
six by six, &c. then it is manifest that

$$x\,B = b$$
$$x^2 D = d$$
$$x^3 F = f$$
$$\&c.$$

Let the values of B, D, F, &c. deduced from these be now substi-
tuted in equation (2) and there will result

$$1 - \frac{b}{x} + \frac{d}{x^2} - \frac{f}{x^3} + \&c. = 0,$$

An equation involving only one unknown quantity x, which is
therefore determined.

The remaining unknown quantities t, t', t'', &c. may now be found
from the series of equations (1) ; accordingly we get

$$t = \sqrt{\frac{r}{x}}, \ t' = \sqrt{\frac{r'}{x}}, \ t'' = \sqrt{\frac{r''}{x}}, \ \&c.$$

The angles of the figure and the radius of its inscribed circle being
known, the manner of determining its sides is sufficiently obvious.
In the diary question, the number of sides being three, D, F, &c. also

d, f, &c. are each $= 0$; therefore $1 - \dfrac{b}{x} = 0$, or, since $b = \sqrt{(rr')}$ $+ \sqrt{(rr'')} + \sqrt{(r'r'')}$,

$$x = \sqrt{(rr')} + \sqrt{(rr'')} + \sqrt{(r'r'')},$$

$$t = \sqrt{\frac{r}{x}},\ t' = \sqrt{\frac{r'}{x}},\ t'' = \sqrt{\frac{r''}{x}}.$$

If the number of sides be four, or six, then $\mathbf{F}$, f &c. are each $= 0$, and x is to be determined from the quadratic equation

$$1 - \frac{b}{x} + \frac{d}{x^2} = 0. \quad \mathbf{w.}$$

II. QUESTION 144, *by Mr.* Will. Massey.

A labourer was hired for a year upon the following conditions, *viz.* for every day he worked he was to receive seven shillings, but for every day he absented himself or was idle, he was to forfeit three shillings : Now at the expiration of the time it was found that he had nothing to receive, his forfeitures being exactly equal to his earnings ; how many days had he worked and how many days had he been idle ?

Answered by Mr. Geo. Anderson.

Let $b = 365 =$ the number of days in a year, $a =$ the number of days wrought ; then $b - a =$ the number of days played ; therefore $7a = 3b - 3a$ per question. Whence a will be found $= 109.5$ days wrought ; and $255.5 =$ the days played.

Additional Solution.

As $10\ (= 7 + 3) : 365 :: \begin{cases} 7 : .7 \times 365 = 255.5, \\ 3 : .3 \times 365 = 109.5. \ \ \mathbf{H.} \end{cases}$

III. QUESTION 145, *by Mr.* Rob. Fearnside.

Two ships A and B, lying west and 'east of each other, are bound to a port D which could not be seen from their stations: The ship A takes her course between east and north $8\frac{5}{14}$ leagues and then discovers the port D, directly to the north of herself and the place where she left B : The ship B proceeded directly east 12.8 leagues, and then despairing to find the port D lay to. The ship A continued her first course till she came within 5 leagues of D, when the two ships and the port were in the same right line. The angle formed at D

by the first and last stations of A is a right angle. Required the distances of the two ships from the port at first, and also the course which the ship A took.
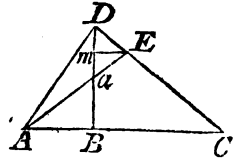
Answered by Mr. Geo. Auderson.

Put A$a = a$, BC $= b$, and DE $= d$. Draw mE parallel to AB, and call it x.

Per similar $\triangle$ s $x : d :: b : \dfrac{bd}{x} = $ CD,

and $x : d :: \dfrac{bd}{x} : \dfrac{bdd}{xx} = $ AC;

Then, by 47 Euc. I. $\sqrt{\dfrac{b^2d^4 - b^2d^2x^2}{x^4}} = $ AD.

Also $\dfrac{bdd - bxx}{xx} = $ AB, $\therefore$ per similar $\triangle$ s$_2$, $\dfrac{bdd - bxx}{xx} : a :: x$

$: \dfrac{ax^3}{bdd - bxx} = $ Ea. By 47 Euol. I. we have

$$\sqrt{\dfrac{aax^6 - 2baax^5 \begin{array}{c}+\ bbaa\\ -\ bbdd\end{array}x^4 + 2hddaax^3 \begin{array}{c}+2bbd^4\\ -2bbddaa\end{array}x^2 \begin{array}{c}-bbd^6\\ +bbd^4aa\end{array}}{bbd^4 - 2bbddxx + bbx^4}} = \text{AD}.$$

Which being made equal to the above value of it, and the equation reduced, we have

$$aax^{10} - 2baax^9 \left\{\begin{array}{c}+\ bbaa\\ -\ bbdd\end{array}\right\} x^8 + 2bddaax^7$$

$$\left.\begin{array}{c}+ bbdd\\ - bbddaa\\ + b^4dd\end{array}\right\} x^6 + \left\{\begin{array}{c}- b^2d^6\\ + b^2d^4a^2\\ - 3h^4d^4\end{array}\right\} x^4 + 3b^4d^4d^6x^2 = b^4d^2.$$

Or in numbers, $x^{10} - 25{\cdot}6x^9 + 105{\cdot}1932x^8 + 640x^7 + 4349{\cdot}031x^6$
$- 64906{\cdot}085x^4 + 18016295{\cdot}945649x^2 = 150135799{\cdot}54708$, &c.

Which solved, is $x = 4$ leagues.

Therefore AD $= 12$ leagues $=$ dist. ship A from D,
    BD $= 9{\cdot}6$ dist. ship B from D,
    BA$a = 30° 28' 53''$ the course of ship A.

The proposer of the question, Mr. Fearnside, gives the course the ship took N.E. by N. $\frac{1}{4}$ 35′ 39″N. but the answer above agrees exactly with him as to the scheme and method.

Remark. To render this question and its solution plain, it may be remarked that, In the triangle ADC, right-angled at D, having drawn AE and the perpendicular DB, there are given A$a = 8\frac{5}{13}$, BC $= 12{\cdot}8$, and DE $= 5$; to find the other parts. H.

IV.   QUESTION 146, *by Mr.* Tho. Grant.

Required the value of x from the equation of $x^x = 123456789$, by a general method which may be applied to any question of the same kind?

Answered by Mr. Tho. Grant.

There is given $x^x = 123456789 = a$.   Quere x?

Let n be assumed near the root x, $z =$ their difference, and $ln =$ the hyperbolic logarithm of n: Then $n \pm z = x$, and (per fluxions) its hyp. log. $= ln \pm \dfrac{z}{n} - \dfrac{z^2}{2n^2} \pm \dfrac{z^3}{3n^3} - \dfrac{z^4}{4n^4} \pm \dfrac{z^5}{5n^5}$, &c. this multiplied by $n \pm z$ gives $nln \pm z \pm zln + \dfrac{z^2}{2n} \mp \dfrac{z^3}{6n^2} + \dfrac{z^4}{12n^3} \mp \dfrac{z^5}{20n^4}$, &c. $= la =$ the hyperb. log. of a.

Put $s = 1 + ln$, $d =$ the difference between la and nln; then $sz \pm \dfrac{z^2}{2n} - \dfrac{z^3}{6n^2} \pm \dfrac{z^4}{12n^3} - \dfrac{z^5}{20n^4}$, &c. $= d$; which by reverting the series becomes $z = \dfrac{1}{s} d \mp \dfrac{1}{2ns^3} d^2 + \dfrac{3+s}{6n^2s^5} d^3 \mp \dfrac{15 + 10s + 2s^2}{24n^3s^7} d^4 + \dfrac{105 + 165s + 60s^2 + 6s^3}{120n^4s^9} d^5$, &c. this wrought in numbers gives $s \pm z \mp x = 8{\cdot}6400268$.

N. B. The above method will find x to any degree of accuracy by only assuming n nearly $=$ to x, and even without that if the logarithm of a and n be made to a sufficient number of places.

Remark. Though the above be a very complete and masterly solution, the method of trial-and-error would sooner approximate to the value of the root. H.

V.   QUESTION 147, *by Mr.* Tho. Williams.

On the 9th of June 1731, there will happen a small eclipse of the moon. The centres of the moon and shadow at the beginning, and the centres of the moon and shadow at the middle of the eclipse, form a right-angled triangle, the sum of whose base and perpendicular will be 83 minutes, and the sum of the base and hypothenuse $= 116$ minutes ; and the diameter will exceed the perpendicular by

4 minutes. Required the diameter of the earth's shadow and the diameter of the moon, and what part of the moon will be covered at the middle of the eclipse ?

<p align="center">*Answered by Mr. W. Grimmet.*</p>

Let $a =$ AC the base, $b =$ AC + AB the sum of the base and perpendicular, $c =$ AC + BC the sum of the base and hypothenuse : Then $b — a =$ the perpendicular; and, per 47 Eucl. I. $\sqrt{(2aa — 2ba + bb)} =$ CB the hypothenuse. Therefore $\sqrt{(2aa — 2ba + bb)}$ $+ a = c$: which equation solved, a will be found $= 54\cdot4985$ the base, and perpendicular $= 28\cdot5015$, to which add 4 gives the moon's diameter $= 32\cdot5015$; and if from the hypothenuse $= 61\cdot5015$, be subtracted the moon's semi-diameter, it leaves CD $= 45\cdot2508 =$ semi-diameter of the earth's shadow ; also from the hypothenuse subtract the base, remains the opaque part of the moon $= 7\cdot0028 = 2$ digits 35' 7''.
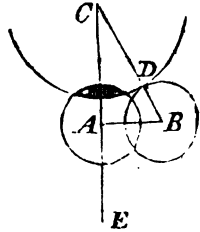
<p align="center">*First Additional Solution.*</p>

This question is the same as, *In a right-angled triangle, having given the two sums of the base and hypothenuse, and base and perpendicular; to determine the triangle.* Of which this is the *Construction.*

The difference between the two given sums will evidently be equal to the difference between the hypothenuse and the perpendicular : Take EA equal to this difference, and EAC $=$ a mean proportional between 2EA and the greater of the above two sums ; and AC will be the base of the triangle. Then the hypothenuse and perpendicular are easily found by subtraction.

<p align="center">*Demonstration.*</p>

Since $CE^2 = (CA + AE)^2 = CA^2 + 2CA \times AE + AE^2$, by Eucl. II. 4,
And AE $=$ CB — BA by the construction,
Also $AE^2 = CB^2 — 2CB \times BA + BA^2$ by Eucl. II. 7,
We have $CE^2 = CA^2 + 2CA \times CB — 2CA \times AB + CB^2 — 2CB \times BA + BA^2$;
Again $CB^2 = CA^2 + BA^2$ by Eucl. I. 47,
Therefore $CE^2 = 2CB^2 + 2CB \times CA — 2CB \times BA — 2CA \times AB$
$\qquad\qquad = (CB + CA) \times (2CB — 2BA)$ (by Eucl. II. 1.)
$\qquad\qquad = (CB + CA) \times 2AE \qquad Q. E. D. \qquad$ **M.**

Second Additional Solution.

Take $BG =$ the sum of the base and perpendicular, and $BD =$ that
of the base and hypothenuse; draw GC to make
the angle $GCB =$ half a right angle, and pro-
duce it to meet a perpendicular, drawn to BD
at D, in F: join FB, and apply $GH = GD$;
draw BC parallel to GH meeting GC in C, and
demit the perpendicular CA; then ACB is the
triangle required.
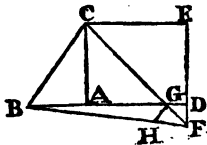
For if CE be drawn parallel to AD, then by
reason of the parallels, and because $GH = CD$ by construction, BC is $=$
$CE = AD$; therefore $BC + BA = BD$ the given sum. Also because
AGC is half a right angle AG is $= AC$; therefore $BA + AC$ is $= BG$
the other given sum. L.

VI. QUESTION 148, by Mr. Tho. Sparrow.

To determine a point in the line joining the centres of the earth
and moon, where a heavy body would be kept in *equilibrio* by their
attraction.

Answered by Mr. John Fearnside.

Two bodies gravitating towards each other, the distances of the
common centre of attraction, are directly as the quantities of matter
in the bodies: Whence the quantity of matter in the earth being,
according to Sir Isaac Newton, to that of the moon, as $39 \cdot 778$ to 1
And the mean distance of the earth and moon $= 1271018757$
English feet, the distance required from the earth 215882 miles,
and from the moon 59018 miles.

Additional Solution.

The above answer to this question is wrong. For, the gravita-
tion towards any body being as its quantity of matter directly and
square of its distance inversely; if e to m denote the proportion of
the quantity of matter in the earth to that in the moon, d the dis-
tance of their centres, and x and z the respective distances from their
centres, of a body placed between them; then the gravitation towards
the earth will be as $\frac{e}{x^2}$, and towards the moon as $\frac{m}{z^2}$: that the body
be at rest between them then, or gravitate equally towards them,
we must make $\frac{e}{x^2} = \frac{m}{z^2}$, and hence $\sqrt{e} : \sqrt{m} :: x : z;$ that is,
the distances from them must be as the roots of the masses, which in
the present case are nearly as $6\frac{1}{4}$ to 1.

Hence $\sqrt{e} + \sqrt{m} : d (= x + z) :: \begin{cases} \sqrt{e} : \dfrac{d\sqrt{e}}{\sqrt{e} + \sqrt{m}} = x, \\[2mm] \sqrt{m} : \dfrac{d\sqrt{m}}{\sqrt{e} + \sqrt{m}} = z. \quad \text{H.} \end{cases}$

VII.   QUESTION 149, by Mr. Rich. Lovatt.

On the 3d of June 1730, the sun's altitude was 33° at the time the hour angle was equal to the co-latitude of the place : what was the time of the day ?

Answered by Capt. Wm. Scarth.

The latitude 26° 33′ N. the hour $4^h\ 13^m\ 48^s$.

Additional Solution.

Here are given z⊙ the co-altitude = 57° (see the spherical tri-angle on page 119) P⊙ the co-declination = 66° 31′, and zP = the angle P. Put a = s. altitude = cos. z⊙, d = s. declination = cos. P⊙, c = its cos. = s. P⊙, and z = cos. PZ or $\angle$ P.

Then, per spherics $\dfrac{a - dz}{c\sqrt{(1 - z^2)}} = z$;

Hence $ccz^4 - ccxx + ddzz - 2adz = -aa$; the root of which equation will answer the question.   H.

VIII.   QUESTION 150, by Mr. John Turner.

There is a meadow in the form of a parabola, the abscissa of which is equal to 40 chains, and the greatest ordinate (that which bounds the meadow) is 72 chains, from whence the latus rectum will be found 32·4 chains.   It is required to inscribe such a parallelogram in this parabola, as that its area may be greater than the area of any other parallelogram that can possibly be inscribed in the said parabola, and to give the analytical investigation of the same ?

Answered by Mr. Geo. Auderson.

Put AD = b, BD = d, the latus rectum = l, and x = bE ? Per conics, $l : d + x :: d - x :$

$\dfrac{dd - xx}{l} = bs.$ By 35 Eucl. I.

$\dfrac{2ddx - 2x^3}{l} =$ area of the parallelogram.
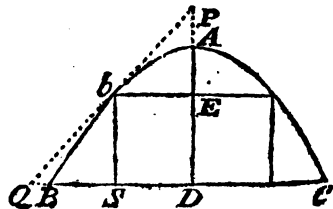
This in fluxions, $\dfrac{2ddx - 6x^2\dot{x}}{l} = 0$, $dd = 3xx$, Hence $x = d\sqrt{\tfrac{1}{3}}$

$= 20{\cdot}7846$, and $\dfrac{2ddx - 2x^3}{l} = 1108{\cdot}512$, &c. square chains $=$

110 acres 3 roods 16 perches.

Additional Solution.

If to the point b there be drawn a tangent meeting the abscissa and ordinate, produced in P and Q: Then, in all curves that are concave towards their axes, it is QB $=$ SB, and therefore DE $=$ EP by similar triangles; but, by the nature of the parabola, AE is $= \frac{1}{2}$EP; therefore AE $= \frac{1}{2}$ED or $= \frac{1}{3}$AD; and consequently the height of the parallelogram ED $= \frac{2}{3}$AD. Also, since the ordinates of a parabola are as the square roots of their abscissas, and the abscissas being as 3 to 1, therefore the ordinates are as $\sqrt{3}$ to 1.

So that AD : AE :: 3 : 1,
and BD : bE :: $\sqrt{3}$: 1. H.

IX. QUESTION 151, *by Mr.* Chr. Mason.

Required the ages of two persons from the following equations

$$aaae + aaeee = 3800000000 = b,$$
$$eeea + esaea = 4067442500 = c;$$

where a represents the age of the older and e that of the younger?

Answer. $\left\{ \begin{array}{l} a = 85{\cdot}379 \\ e = 79{\cdot}899 \end{array} \right\}$ their ages.

Additional Solutions.

1. Of the two given equations $\left\{ \begin{array}{l} a^3e + a^2e^3 = b \\ a^3e^2 + ae^3 = c \end{array} \right\}$, the latter being multiplied by a, and the former taken from the product, there remains $e^2a^4 - ea^3 = ca - b$; hence $e = \dfrac{a \pm \sqrt{(a^2 + 4ac - 4b)}}{2a^2}$:

this being substituted in one of the original equations, there will result an equation with only one letter in it, and which from it may be found. H.

2. Take two near values of a and e, as 85 and 80, and call the differences x and z; then $a = 85 + x$, and $e = 80 - z$: substitute these in the two original equations, neglecting all the terms which rise to above two dimensions, and there will result two quadratic equations, from which x and z may be found. H.

3. Put $e = xa$, x being a new unknown quantity; then by sub-stitution

$$xa^4 + x^3a^5 = b, \qquad x^5a^4 + x^2a^5 = c;$$

from these equations we readily find

$$a^5 = \frac{bx^2 - c}{x(x^4 - x)}, \qquad a^4 = \frac{cx - b}{x^4 - x},$$

and by division $\qquad a = \dfrac{bx^2 - c}{x(cx - b)},$

by comparing this with the values of a^4 or a^5 we find.

$$x^3(cx - b)^5 = (x^3 - 1)(bx^2 - c)^4$$

or, putting $\dfrac{c}{b} = m$, $\dfrac{b}{c} = n$, on account of b and c being expressed by large numbers,

$$(x^2 - m)^4 (x^3 - 1) = cm^4(x - n)^5 x^3.$$

From this equation the value, or values of x may be found by the usual methods of approximation, and thence the values of a and e from the equations

$$a = \frac{bx^2 - c}{x(cx - b)}, \qquad e = xa. \quad \text{L.}$$

THE PRIZE QUESTION.

At the time of the last eclipse of the moon, when lunar affairs were of course the subject of general conversation, the following ques-tion was proposed. Suppose that A, B, and C make a stock of 22650 hundred weight, and it sold at the moon for 2336193l. 9s. 7$\frac{1}{6}$d. being sold at a hundred times more per ton than it cost; and sup-pose the price of A's stock in pounds to be the square root of the stock of B; and the stock of A and B, the square root of the stock of C. Quere each man's stock and gain, and the weight of the goods at the moon; and because if the same goods, at the same price, had been sold at the globe of Jupiter, the amount had been 9108924l.? Quere the weight at his body also?

Answered.

This question may admit of some differences in the answer as the proportions vary in authors. I shall therefore first give you the proposer Mr. *Tho. Pointin*'s answer.

' The distance of the earth and ☉ 80015040 miles.
From the centre of ⊖ to ☾ — 236350
From centre ☉ to ☾ surface — 80251390
The weight of a body at the ☉ and ☾ as 1 to ·516; the force of the ⊖ at ☾ ·516 — ·000284; = ·515716 × 22650 = 11680·96

the weight at the ☾ ; by which the prime cost is 2l. per hundred ;. for ·515716 × 100 × 22650 = 1168096·74) 2336193·48. (2. Let 2a =A's cash, 4aa = B's, and 16a⁴ + 8a³ + aa = c's stock. Therefore 16a⁴ + 8a³ + 5aa + a = sum. A = 6 price 12 lib. B 150, c 2250, and total stock = 22650. The total gain 2290893, of which A's gain = 606·8592 lib. B's = 14564·6208, c's = 2275722 : the weight at the earth and Jupiter as 1 to 2·0108. So the weight will be 4554462 = 2033 ton, 4 cwt. 94 lb. Mr. *Grimmet*, Mr. *Sidebottom*, Mr. *Bullman*, Mr. *Fearnside*, Mr. *Pilgrim*, and some others, make the weight at the moon 567 tons, at Jupiter 2210 tons. Mr. *Richards*, Mr. *Fairchild*, Mr. *Hale*, Mr. *Lowe*, Mr. *Mason*, Mr. *Grant*, and some others, 386 ton at ☾, and 15071 at Jupiter : several of whom have wrought this question very curiously, and in an algebraic method, but my room will not permit inserting them.

———•••◐◐●●——

Questions proposed in 1731, and answered in 1732.

I. QUESTION 152, by Mr. Tho. Williams.

A ball projected at the elevation of 31° above the horizon, from the top of a lofty tower, in 9¼ seconds fell 2000 feet from the base. Query, the tower's height?

Answered by Mr. Rob. Fearnside.

In the same time the ball describes BE, another body let fall from E would fall the space EF, and consequently in 9″ 10‴ time, when the ball is at H, a heavy body will describe CH = 1351·4 feet. Then as cosine 31° : BD = 2000 feet :: sine 31° : CD = 1201·7 feet; consequently the tower's height = CH — CD = 149·72 feet = AB.
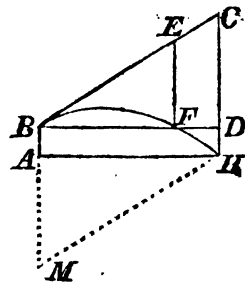
The same answered by Mr. Wm. Grimmett.

Let BA represent the pillar, AH the distance the ball falls from the base of pillar equal to 2000 feet. Let CH be perpendicular to AH;

BC the line in which the projectile is directed, in which it would move equal spaces in equal times, were it not deflected downwards by the force of gravity. 'Tis known all projectiles will (rejecting the resistance of the medium) describe parabolas: therefore BC is the tangent of that parabola, BD is parallel to AH the horizon: draw MH parallel and equal to BC: consequently MH is an ordinate to the diameter BM. Then in the right-angled triangle BDC there is sufficient given to find the sides. And since the velocities of all projectiles in the several points of the curve, are as the lengths of the tangents to the parabola in those points, intercepted between any two diameters; therefore BC = 2333·33 divided by 9⅙ (= number of seconds the ball was in motion) gives 254·54 the velocity in a second; and the square of the velocity divided by 16¹⁄₁₂ feet, the space a body describes in a second by gravity, gives the parameter to the diameter BA. That is (per conics) the rectangle under the intercepted part of the diameter taken from its origin B towards A, and the parameter, is = the square of its correspondent semi-ordinate: whence the parameter is = 4028·43; and the square of MH divided by the parameter, gives BM = 1351·5. But BM — AM = BA = 149·5 the height of the tower.

II. QUESTION 153, by Mr. Geo. Auderson.

Mr. Euclid Speidell, in his Geometrical Extractions, p. 59, says, if upon C, one end of the base CA, of the triangle ABC, you raise a perpendicular CD, equal to the perpendicular BP, let fall from B to CA, and join DA; that the right line CE bisecting the angle ACD, being produced, shall cut DA in such a point F through which, if the right line GH be drawn parallel to the base CA, the part GH intercepted between the two sides BC and BA, shall be the side of the square made in the aforesaid triangle. Quere, the geometrical demonstration?

Answered.

Produce HE to L, and draw FM and GK parallel to PB.

   (1) BP : BN :: CP : GN ⎫ per figure,
   (2) BP : BN :: PA : NH ⎭
Per 11. 5 Euc. (3) CP : GN :: PA : NH.
Per altern. (4) CP : PA :: GN : NH.
Per comp. (5) CA : PA :: GH : NH.
Per altern. (6) CA : GH :: PA : NH.
Per steps 2. 6. (7) BP : BN :: CA : GH.
But the ∠MCF per hypothesis is ½ a right angle, and CMF is a right angle, and consequently CFM is ¼ a right angle, and therefore CM or LF = FM = GK. Moreover DC = BP per hypothesis, and DL = BN per construction. Therefore
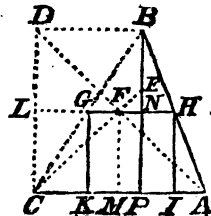
Per. step 7.   (8) DC : DL :: CA : GH.
But per fig.   (9) DC : DL :: CA : LF.
Then per 11 }(10) CA : GH :: CA : LF.
   E. 5.   }
Therefore GH = LF = GK.   Q. E. D.

Answered by Mr. Will. Grimmett.

The geometrical demonstration of this problem may be easily seen from the proposer's own scheme, without any previous construction, (viz). Since there cannot be but one square inscribed in that position, and that LC is equal to LF each subtending an angle of 45° (per prob.): therefore if we prove that LF is = GH universally, it is demonstrated; for LC is equal to GH (which is supposed to be a side of a square.)

Demonstration. From 37 Euc. 1, the triangles DCB and DAB are equal; and if an infinite number of right lines as LH be drawn parallel to the common base DB, the parts of those lines which are intercepted by the respective sides of the triangles, will also be equal, that is LG = FH ; which is Cavalerius's method of proving the triangles to be equal (which is the method of indivisibles;) therefore if to both be added or subtracted (as the case requires) GF, LF = GH. *Q. E. D.*

Additional Demonstration.

This construction may be otherwise demonstrated thus :
As DC or BP : CA :: DF : FA (Eucl. VI. 3.) :: DL : LC or GK
  (Eucl. VI. 2), but BP : CA :: BN or DL : GH ; ∴ GK = GH.

Why the proposer of the question calls *Speidell* by the name of *Euclid,* I know not. His name was *John,* as appears by the book, which was published in the year 1617. H.

III. QUESTION 154, *by Mr.* John Turner.

Let a stone be whirl'd round in a sling, whose length is equal to 33 inches, performing each revolution in the time of two thirds of one second. Required the ratio of the centrifugal force to the weight of the stone? That is, the ratio of the tension of the string whirled round in this manner, to the tension arising from the force of the same weight hanging freely, and without motion?

Answered by the Proposer.

As 113 : 355 :: 66 : 207·35 inches, the arch described in ⅔ of 1″; and 207·35 squared and divided by 66, the double length of the

sling, is $=$ 651·03 the centrifugal force. But the descent of a heavy body in $\frac{2}{3}$ of $1''$ is $=86·208$ inches : whence the centrifugal force is to the force of gravity as 651·4 to 85·7 or as 7·6 to 1.

The same answered by Mr. John Ommanney.

The stone, if let fall out of the sling when at rest, by the force of gravity would descend $85\frac{1}{4}$ inches; which call b. Now let F be the force of the stone or tension of the string when at rest; and E that when in motion. The lengths gone over are always propor-tional to the force impressed. Therefore $F : E :: b : \dfrac{Eb}{F}$ the space descended through with the force E in the same time. Let $c =$ the cir-cumference of the sling $= 207·344$, and $d =$ diameter $= 66$; then, by Sir Isaac Newton, $\dfrac{cc}{d} = \dfrac{Eb}{F}$. Whence $F : E :: db : cc$. That is as 5659·5 to 42991·9, or as 1 to 7·596. $Q. E. I.$

IV. QUESTION 155, by Mr. John Turner,

Suppose the earth's radius $=$ 6982000 yards, and that there is a mountain upon its superficies of such an height, and that a clock when on the earth's surface, shall point out equal time ; but when carried to the top of the mountain, shall be so retarded as to err 2 minutes every day. 'Tis required to give that mountain's elevation?

Answered.

The number of oscillations in each place being equal, will be as the time in one place to the time in the other, viz. as 1440 to 1442. Let $n =$ earth's radius, and $a =$ radius and the height of the moun-tain together ; the force of gravity being inversely as the square of the distance from the centre, we shall have $1442n = 1440a$, and $\dfrac{1442n}{1440}$ $= 6991698 = a$; from which subtracting the radius of the earth $=$ 6982000, remains the mountain's height $= 9698$ yards $= 5\frac{1}{2}$ miles 18 yards.

Answered by Mr. Geo. Anderson,

The length of the pendulum vibrating seconds on the mountain's top will be $= 39·09$ inches; therefore by Huygen's proportions the space descended in a second of time on the top of the mountain will be 192·9009 inches ; and the space descended in a vibration of the clock is 193·4437 inches : And the gravity being defined by the space passed over in a second of time. From corol. 2, prop. 5,

theor. 5, Principia Philosophiæ ; put $r =$ earth's radius, $x =$ mountain's height.

$$r^2 + 2xr + x^2 : r^2 :: 193\cdot44 : 192\cdot9.$$

And therefore $x = \cdot0014059r = 353375\cdot7768$ inches, or $9815\cdot99$ yards, or $5\cdot5773$ miles.

Additional Solution.

Universally, the lengths of pendulums being directly as the force of gravity drawn into the squares of the times of vibration, and the pendulum being of the same length in both cases, we shall have $ft^2 = \text{F}\text{T}^2$; putting f and t for the force of gravity and time of vibration at the earth's surface, and F and T for the same at the top of the hill. But the force of gravity is inversely as the square of the distance from the earth's centre; that is f is as $\dfrac{1}{r^2}$, and F as $\dfrac{1}{(r+x)^2}$, putting r for the earth's radius, and x for the height of the hill ; then the above equation becomes $\dfrac{t^2}{r^2} = \dfrac{\text{T}^2}{(r+x)^2}$ or $\dfrac{t}{r} = \dfrac{\text{T}}{r+x}$. Again, the time of a vibration being inversely as the number of vibrations in a given time, and the number of vibrations being as 1440 to 1438 ; the same equation will become $\dfrac{1}{1440r} = \dfrac{1}{1438 \times (r+x)}$. Hence $x =$ $\dfrac{2r}{1438} = \dfrac{r}{719} = 9710\cdot7$ yards $=$ the height of the mountain required.

Corollary.

Hence it appears in general, that if l be the time lost in the time t ; then 24 hours $- l : l :: r : x$; or, because 24 hours $- l$ is nearly $= 24$ hours, it will be nearly as 24 hours $: l :: r : x$; that is, the whole time is to the time lost, as the radius of the earth to the height of the hill. H.

V. QUESTION 156, by Mr. William Grimmett.

Required the greatest parallelogram that can be inscribed in an ellipsis, whose transverse is 10, and conjugate 8. And supposing the ellipsis to revolve upon its transverse axis, required also the greatest cylinder that can be inscribed in the spheroid that is generated by the revolving ellipsis ; and to shew the analytical investigation of the same ?

Answered by Mr. John Turner.

Let $\text{SD} = t = 5$; $\text{BS} = c = 4$; $\text{S}t = a$; $\text{D}t = t + a$; and $\text{A}t = t - q$. Then we have

As $tt : cc :: tt - aa : \dfrac{cctt - ttaa}{tt} =$

square of cx. Ergo $tx = \sqrt{\dfrac{cctt - ccaa}{tt}}$

and $cx = \sqrt{\dfrac{4cctt - 4ccaa}{tt}}$. This

multiplied by $vt = 2a$, makes the area of the rectangle $ymxc = 2a \times$

$\sqrt{\dfrac{4cctt - 4ccaa}{tt}} = \sqrt{\dfrac{16ccttaa - 16cca^4}{tt}}$: But this, by the question, must be a maximum; therefore the fluxion of it which is

$\dfrac{32\dot{c}cttaa - 64cca^3\dot{a}}{2\sqrt{(16ccttaa - 16cca^4)}} = 0.$ From hence $64ccaa = 32cctt$: And

$a = t\sqrt{\frac{1}{2}} = 3\cdot535$: And $2a = vt$ is $= 7\cdot070$, and ym or $xc = 5\cdot657$. Again, for the greatest cylinder, supposing it to revolve about the

transverse axis, we found above $cx = \sqrt{\dfrac{4cctt - 4ccaa}{tt}}$. Ergo cx

squared is $= \dfrac{4cctt - 4ccaa}{tt}$; this multiplied by $vt = 2a$ must, per

question, be a maximum. The fluxion of $\dfrac{8cctta - 8cca^3}{tt}$ is

$8cctt\dot{a} - 24ccaa\dot{a} = 0.$ Hence we have by division and transposition $a = t\sqrt{\frac{1}{3}} = 2\cdot887$: And $vt = 2a = 5\cdot774$ the cylinder's altitude, and xc or my the diameter of the base $= 6\cdot532.$ **Q. E. I.**
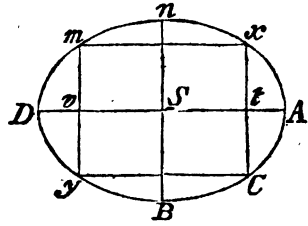
The proposer's answer.

Put $d = $ AD, $c = n$B, $x = $ DV, $y = yv$. Then $2y$ and $d - 2x$ the sides of the parallelogram; the rectangle of which is $2yd - 4xy$, the fluxion of which, if a maximum, is $2d\dot{y} - 4\dot{x}y - 4x\dot{y} = 0$, or $d\dot{y} - 2\dot{x}y - 2x\dot{y} = 0$. By the property of the ellipse $dd : cc :: dx$

$- xx : yy$. Ergo $y = \sqrt{\dfrac{dccx - ccxx}{dd}}$. In fluxions $\dot{y} =$

$\dfrac{dcc\dot{x} - 2ccx\dot{x}}{2d\sqrt{(dccx - ccxx)}}$. Which substituted in the former, we have

$\dfrac{dcc\dot{x} - 2ccx\dot{x}}{\sqrt{(dcc x - ccxx)}} - 2\dot{x}\sqrt{\dfrac{dccx - ccxx}{dd}} - x \times \dfrac{dcc\dot{x} - 2ccx\dot{x}}{d\sqrt{(dccx - ccxx)}} = 0.$

Which reduced, then $xx - dx + \frac{1}{4}dd = -\frac{1}{8}dd$.

Hence $x = \frac{1}{2}d - \dfrac{d}{\sqrt{8}} = 1\cdot4645 = $ DV: from whence mx will

be found $= 7\cdot071$, and $my = 5\cdot6568$: and the area of the greatest parallelogram $= 39\cdot999999.$ **Q. E. I.**

Additional Solution.

In the solution of question 74, at page 96, it is proved that CE $=$ CA$\sqrt{\frac{1}{3}}$, (*see the fig. to that solution*) and EF $=$ CD $\sqrt{\frac{2}{3}}$; CE being half the height, and EF half the diameter of the cylinder, when CA is the semi-transverse, and CD the semi-conjugate axe.

Again, supposing CE and EF to be the halves of the length and breadth of the greatest inscribed parallelogram, by page 201 Simpson's Geom. 2d edit. CE is $=$ $\frac{1}{2}$CT, and EF $=$ $\frac{1}{2}$CX; but, by the nature of the ellipse, CE is : CA :: CA : CT or 2CE ; hence CE $=$ CA $\sqrt{\frac{1}{2}}$. And in like manner EF $=$ CD $\sqrt{\frac{1}{2}}$.

Corollary.

Hence, if t and c denote the whole transverse and conjugate axes. Then $t\sqrt{\frac{1}{2}}$ $=$ length of the greatest rectangle,

$\qquad c\sqrt{\frac{1}{2}}$ $=$ breadth of the same,

$\qquad \frac{1}{2}ct$ $=$ area of the same.

And $t\sqrt{\frac{1}{3}}$ $=$ length of the greatest cylinder,

$\qquad c\sqrt{\frac{2}{3}}$ $=$ diameter of it,

$\qquad \dfrac{2c^2 tn}{3\sqrt{3}}$ $=$ content of it, putting $n =$ ·78539, &c. N.

VI. QUESTION 157, *by Mr.* Christ. Hale.

On the evening of the 9th of May, 1730, a person viewing the heavens, in latitude 29° 22′ N. observed a star whose altitude was equal to the hour-angle from noon, and also equal to the azimuth. Query, the name of the star ?

Answered by Mr. Geo. Auderson.

In contemplating on the solution of the 159th question, I hit upon the following lemma, which I find to be so useful in solving astronomical questions of that kind, that I believe there can no such question be proposed, but it will be found an useful instrument in solving it.

In the spherical triangle let P be the north pole, z the zenith, ⊙ the place of the sun : and

$$\text{Put} \begin{cases} 1 = \text{radius,} \\ a = \text{tangent of } z\text{P,} \\ s = \text{cosine of } z\text{P,} \\ d = \text{right sine } \odot \text{P,} \\ e = \text{cosine of } \odot \text{P,} \\ g = \text{cosine of } \odot z, \\ x = \text{cosine of } \odot \text{PZ} \end{cases}$$
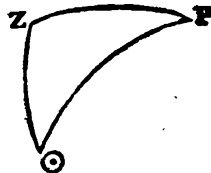
LEMMA:
$(e + dax) \times s = g.$

Corollary.

In question 157 there is given the latitude of the place, and the altitude equal to the hour from noon and to the azimuth, to find the latitude of the star (or declination with respect to the sun); therefore in the room of g in the lemma put $\sqrt{(1-xx)}$ ($=$ sine of the hour) and there will be had $(e+dax) \times s = \sqrt{(1-xx)}$, from the equality of the hour and altitude. And now to answer the other parts of the question, because the azimuth is equal to the hour, therefore the same x still represents the cosine of the angle $\odot$ ZP, and the theorem must be so transposed that $\odot$ ZP may be the angle contained: wherefore since ZP is one of the containing sides of the angle $\odot$ ZP, as well as of the angle $\odot$ PZ, therefore a and s may still retain their former office, but for d must be wrote $\sqrt{(1-gg)}$, and for e put g, and for g write e, and for e put x, and then the lemma will stand thus, $(x + ax\sqrt{(1-gg)}) \times s = e$. But because $\sqrt{(1-gg)} = x$, the theorem will be $sx + saxx = e$; therefore from the two equations $(e + ax \sqrt{(1-ee)}) \times s = \sqrt{(1-xx)}$ and $x + saxx = e$, exterminate e you will have this equation $s^8a^6x^{12} + 4s^8a^7x^{11}$

$$+ 8s^8a^6x^{10} + 12s^8a^5x^9 \left.\begin{matrix} - 2s^4a^4 \\ + 14s^8a^4 \end{matrix}\right\} x^8 \left.\begin{matrix} - 4s^4a^3 \\ + 12s^8a^3 \end{matrix}\right\} x^7 \left.\begin{matrix} + 2s^4a^4 \\ - 4s^8a^4 \\ + 8s^8a^2 \end{matrix}\right\} x^6$$

$$\left.\begin{matrix} + 4s^4a^3 \\ + 4s^4a \\ + 4s^8a \\ - 8s^6a^3 \end{matrix}\right\} x^5 \left.\begin{matrix} + 2s^4 \\ - 4s^6a^2 \\ + s^8 \\ + 1 \end{matrix}\right\} x^4 - 4s^4ax^3 \left.\begin{matrix} - 2s^4 \\ - 2 \end{matrix}\right\} x^2 + 1 = 0.$$ Which in

numbers reduced will give the cosine of the hour.

The proposer's answer.

Let $\sqrt{(1-aa)} =$ sine of Z$\odot$ or P$\odot$, $b =$ sine of ZP comp. lat. 60° 38′, $n =$ sine lat. 29° 22′, $a =$ sine of the altitude.

As $\sqrt{(1-aa)} : a :: b : \dfrac{ab}{\sqrt{(1-aa)}} =$ sine $\angle$Z$\odot$P and $1 - \dfrac{b^2a^2}{1-aa} =$ the square of its cosine; $1 - aa : 1 :: 1 - n : \dfrac{1-n}{1-aa}$

$=$ versed sine $\angle$Z$\odot$P, and $\dfrac{n-aa}{1-aa} =$ cosine $\angle$Z$\odot$P. Hence

$\dfrac{nn - 2naa + a^4}{1-aa} = 1 - aa - a^2b^2$. Reduced we have $a = \cdot7493411$

$=$ sine of 48° 33″ $=$ to the altitude and latitude of Perseus's right side.

Additional Solution,

In this question is given ZP, the $\angle$z $= \angle$P $=$ comp. $\odot$z or $\odot$P, the triangle being isosceles. Then supposing a perpendicular

demitted from $\odot$ upon and bisecting zp, and thereby dividing $\odot$ zp
into two equal right-angled triangles ; put $a =$ the tangent of the
base of each or half pz the given colatitude, and $x =$ sine of the
hypothenuse $\odot$ P or $\odot$ z $=$ cos. $\angle$ P or z; then, by right-angled
triangles, (radius $=$) $1 : x :: \dfrac{x}{\sqrt{(1-x^2)}}$ $(=$ tang. $\angle$ P or z) :

$\dfrac{x^2}{\sqrt{(1-x^2)}} = a$; hence $x^4 = a^2 - a^2 x^2$, and $x = \sqrt{(a\sqrt{(\frac{1}{4}a^2+1)} - \frac{1}{2}a^2)}$

$=$ the cosine of the declination.   H.

<div align="center">VII.   QUESTION 158, by Mr. John Fearnside.</div>

There is a garden of a triangular form, the three sides are orna-
mented with the statues of Bacchus, Apollo, and Hermes; there are
also straight walks between each of the statues; the walk from
Bacchus to Hermes is 216·4074 feet, that from Apollo to Bacchus
192·1306 feet, and the walk from Apollo to Hermes is 209·694 feet.
The length of that side of the garden in which the statue of Hermes
is placed is 363 feet, and one of the angles adjacent to this side is 55°.
Required the two unknown sides of the garden, one of the walks
being perpendicular to one of them?

<div align="center">Answered by Mr. J. Fearnside.</div>

Let DC $= m$, AB $= b$, AH $= c$, HB $= a$, sin. $\angle$ D $= s$, its cos. $= l$,
sin. $\angle$ CAH $= r$, sin. HBA $= u$, its cos. $= t$, and

BF $= x$. The will sin. $\angle$ F $= \dfrac{b}{x}$, and its cos. $=$

$\sqrt{\dfrac{xx-bb}{xx}}$;   sin. c $= \dfrac{bl}{x} + s\sqrt{\dfrac{xx-bb}{xx}}$;

HC $= \dfrac{cr}{\dfrac{bl}{x} + s\sqrt{\dfrac{xx-bb}{xx}}}$; sin HBD $= \dfrac{ub}{x} + t \times$

$\sqrt{\dfrac{xx-bb}{xx}}$; and HD $= \dfrac{uba}{sx} + \dfrac{ta}{s}\sqrt{\dfrac{xx-bb}{xx}}$.

Whence this equation $\dfrac{cr}{\dfrac{bl}{x} + s\sqrt{\dfrac{xx-bb}{xx}}} + \dfrac{aub}{sx}$

$+ \dfrac{ta}{s}\sqrt{\dfrac{xx-bb}{xx}} = m$. Which reduced gives $x = 233{\cdot}10474$.

Therefore DF $= 412{\cdot}5$, CF $= 427{\cdot}4$, and the area 1 ac. 1 r. 20 p.

   This question being unlimited, as not determined on which wall
Bacchus or Apollo stands, will admit of another answer.
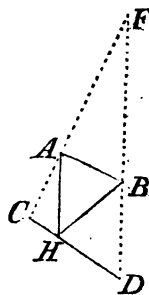
Remark. In this question it is required, about a given triangle to circumscribe another triangle such, that one of its angles, and one of the sides about that angle, shall be each of a given magnitude ; and the side opposite to the given angle perpendicular to a side of the given inscribed triangle.

And hence if upon the given line BH be described the segment of a circle capable of containing the given angle D, the problem is reduced to this : To apply a line CD of a given length between the circumference of that circle and the line CF given in position, and so as to pass through the given point H. H.

Solution of the problem indicated in the preceding remark.

Suppose the thing done. Let o be the centre of the circle ; join HO, draw the radius OP parallel to HD, and PE perpendicular to HO. The chord HD is twice the cosine of the angle POE, to the radius OP, and OE is the cosine of the same angle, therefore $OE = \frac{1}{2}HD$. Take OI $= \frac{1}{2}$ the given line CD, then I will be a given point and $EI = \frac{1}{2}CH$. Through o drawn NOL parallel to CF. Draw MHN and IL perpendicular to HI, the former meeting CF, NL in M and N, the latter meeting NL in L ; and produce PE to meet NL in K.
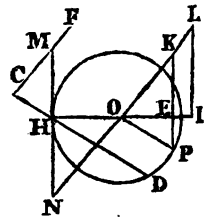
Because of the similar triangles HCM, POK, and the parallel lines IL, EK, HN, we have

$$HM : HC \text{ or } 2IE :: KP : PO \text{ or } OH,$$
$$2IE : 2LK :: OH : ON ;$$

therefore $HM : 2LK :: KP : ON :$

Hence $2LK \times KP = HM \times ON.$

Now from the disposition of the lines, it is manifest that HM and ON are both given in magnitude ; and the point L and the line OL given by position, this last making with PK a given angle ; therefore the point P is evidently in a given hyperbola of which OL and LI are the asymptotes. And this curve being described, the point or points in which it cuts the circle will determine the point P, and consequently the position of CD which is parallel to OP.

The composition is easily deduced from the analysis, and for the sake of brevity is omitted. w.

VIII. QUESTION 159, *by Mr.* John Fearnside.

What time of the day is the hottest at York on the 8th of June, supposing it to be as the sine of the sun's altitude, and the time of his continuance above the horizon ?

Answered by Mr. Auderson.

Given the latitude, the sun's declination, and the sun's heat, as the sine of his altitude and continuance above the horizon (viz. as their product) for so I take it, and was so informed by E. Stone, who proposed the same question in his preface to de l'Hospital's fluxions.

Lemma $(e + dax) \times s =$ sine sun's altitude, as in question 157.

But $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 +$ &c. is the arch belonging to x,

and $b - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 - \frac{5}{112}x^7$ &c. (putting $b = \frac{1}{4}$ of the circumference) is the complement of it, and therefore the hour from noon: Put $r =$ the time (viz. the length of the arch of the equinoctial) passed over from sun-rising to noon : Now it is manifest the aforesaid phænomenon will not happen before the sun's arrival at the meridian, therefore $r + b - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 - \frac{5}{112}x^7 -$ &c.

is the sun's continuance above the horizon, and this multiplied by $se + sdax$ the sine of the altitude, and thrown into fluxions and reduced gives $2sdax + \frac{1}{3}sex^3 + \frac{2}{3}sdax^3 + \frac{1}{3}sex^4 + \frac{2}{20}sdax^5 + \frac{5}{16}sex^6 +$

$\frac{5}{14}sdax^7$ &c. $= sdar + sdab - se - m$. And therefore $x = \dfrac{m}{2sda} -$

$\dfrac{m^2}{2se \times (2sda)^2} + \dfrac{3ees - 4da}{4da} \times \dfrac{m^3}{(2sda)^3} -$ &c.

The proposer's answer.

Let $m =$ the rectangle of the sine com. lat. and co-declination, $p =$ sun's alt. at 6, $n =$ the length of the semi-diurnal arch, and z the length of the arch from noon. The cosine of the hour angle

from noon is $= 1 - \dfrac{zz}{2} + \dfrac{z^4}{24} - \dfrac{z^6}{720}$ &c. and, putting $m + p = a$, the

sine of the sun's altitude will be $= a - \dfrac{mzz}{2} + \dfrac{mz^4}{24} - \dfrac{mz^6}{720}$, &c.

Which multiplied by $n + z$, and thrown into fluxions, &c. gives

$mnz + \dfrac{3m}{2}z^2 - \dfrac{mn}{6}z^3 - \dfrac{5m}{24}z^4 +$ &c. Reversed z will be $= \cdot 50614$

(the radius being 1) consequently the hour is $1^h\ 56'$ afternoon.

Additional Solution.

Put s and $c =$ the sine and cosine of PZ, d and $e =$ sine and cosine of P$\odot$, $m =$ the semi-diurnal arc, $x =$ the hour arc or $\angle$ P, $y =$ its sine, and $z =$ sine of the altitude or cosine of Z$\odot$.

Then $mz + xz =$ a maximum, per question, or $m\dot{z} + x\dot{z} + \dot{x}z = 0$.

But $z = ee + ds \sqrt{(1 - y^2)}$, or $\dot{z} = \dfrac{- ds y \dot{y}}{\sqrt{(1 - y^2)}}$; and $\dot{x} = \dfrac{\dot{y}}{\sqrt{(1 - y^2)}}$.

Wherefore $\dfrac{- dsmy\dot{y}}{\sqrt{(1 - y^2)}} - \dfrac{dsxy\dot{y}}{\sqrt{(1 - y^2)}} + ds\dot{y} + \dfrac{ce\dot{y}}{\sqrt{(1 - y^2)}} = 0$,

or $ce - dsmy - dsxy + ds \sqrt{(1 - y^2)} = 0$,

and hence $x = \dfrac{ce - dsmy + ds \sqrt{(1 - y^2)}}{dsy}$.

Now this equation may be resolved after several different manners. One method is to substitute in it the value of x in terms of y in an infinite series, and this will be like the two original solutions : Or a finite approximate value of x in terms of y may be substituted for it, and then the resulting equation will be finite : Or some method may be used for approximating to the values of x and y immediately from the equation as it stands above, which will be easiest and soonest done; and the best method for that purpose seems to be this; by a few trials find $y = \cdot 5$ nearly, which substituted in the above form, a near value of x will be found ; from a table of sines, to this value of x take the correspondent value of y, which written in the above form again, there will be had a nearer value of x ; and so on as far as necessary. H.

See also question 241.

IX. QUESTION 160, *by Mr.* Christ. Mason.

A surveyor wants to determine the distances from a certain station to two objects, 40 chains asunder, and the angle which they subtend 30° : accordingly he measures 23 chains directly south of the station, and then finds the objects exactly west, and in the same right line. What are the distances from the said station.
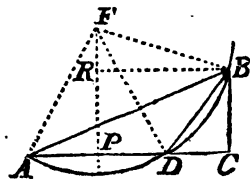
Answered by the author Mr. Mason.

Let AD $= b = 40$, BC $= c = 23$, $d = s$. $\angle$ ABD $= 30°$, $a =$ DC $= ?$

Per 47 E. 1, $\sqrt{(b^2 + 2ba + aa + cc)}$ $=$ AB; for which substitute y : then

$y : 1 :: c : \dfrac{c}{y} = s$. $\angle$ BAC; and $d : b ::$

$\dfrac{c}{y} : \dfrac{bc}{dy} =$ DB; per 47 E. 1, $\dfrac{bbcc}{ddyy} - cc$

$= aa$ i. e. $bbcc - ccddyy = ddyyaa$:

for yy put its value. Then $a^4 + 2ba^3 + bbaa - 2bca = \dfrac{bbcc}{dd} + c^4$

$- bbcc$: which reduced gives $a = 18\cdot3$, AB $= 62\cdot672$, DB $= 29\cdot392$.

<div align="center">Answered by Mr. John Bulman.</div>

On the given segment AD $= 40$ chains (per 33 Eucl. 3.) describe a segment of a circle which shall contain the given angle ABD $= 30^\circ$. And from F the centre thereof let fall the perpendicular FP upon AD; make PR $= 23$, and draw RB parallel to AD meeting the circle in B; and let fall the perpendicular BC. Then the lines FB, FD, FA will all be equal as being radius's of the same circle.

The angle AFD is double the angle ABD (per 20 E. 2); therefore the triangle AFD is equilateral, and consequently equiangular, each side being 40 chains, and each angle 60°. Then $FD^2 - PD^2 = FP^2$ (per 47 E. 1); whence FP $= 34\cdot641$, and FP $-$ BC $=$ FR $= 11\cdot641$. Again, $FB^2 - FR^2 = RB^2$; whence RB $= 38\cdot2686$, and RB $-$ PD $=$ DC $= 18\cdot2686 =$ the distance of the nearest mark from the end of the first line: Hence BD may esaily be found $= 29\cdot3628$ chains, and BA $= 62\cdot6438$. Q. E. I.

<div align="center">X.　QUESTION 161, by Mr. Tho. Pointon.</div>

> In th' Atlantic ocean an island is found,
> Neither oblong nor square, but perfectly round;
> In the centre of which a tower's erected,
> From whence, with much ease, their foes are detected :
> For if Neptune but smile, and the sky be serene,
> Four leagues * from the shore is the horizon seen ;
> We being at anchor, o'er the top o'th' mast
> From the top o'th' tower a line being past,
> Did bound the utmost stretch of sight at last.　·
> From thence we set sail, and I'll tell you in short,
> The distance † we ran, to arrive at our port :
> And also our course ‡ as corrected at noon,
> Perform'd with great care by the height of the sun :
> Which, to my surprize, neither higher nor lower,
> Was cut in two halves by the fore-mentioned tower.
> The height of our mast from the water's here shown,
> And whatever else is most fit to be known § ;

---

* Note, $69\frac{1}{2}$ *miles to a degree, three of which are a league.*

† *Our distance* $\frac{1}{2}$ *of a degree.*

‡ *Our course N. by E.* 2° $45'$ $E.$ *or* 14° *from N. to E.*

§ *The height of the mast from the water* 45 *feet, Lat.* 64° $20'$ $N.$ *of the tower* 63° $56'$. *The ship's distance from the shore is* $1\cdot263244$ *leagues.*

Which being supposed, what I want to be found,
Is the tower's just height, and land's compass round.
Besides, when you're got in this fig'ring strain,
Let's know the square miles the whole doth contain.

Answered by the Author.

The longitude is 20′ 23″ 2‴ 24⁗; from which, and two latitudes, is found the tower's distance 26′ 12″. The arch of distance between the ship and horizon is 7′ 5″ 16‴ 48⁗ + . 26′ 12″ = 33″ 17″ 16‴ 48⁗. By which 326·15264 yards is the tower's height, and 56·6202 leagues the island's compass ; also the area 2193·2553 square miles.

It is not easy to say what the proposer's real meaning is in this question, but we have printed it, and the solution, exactly as they stand in the original. L.

XI. QUESTION 162, by Mr. C. Mason.

It is required to find two such numbers, viz. A and B, that the sum of the aliquot parts in A may be equal to $\frac{2}{3}$ of B, and the sum of the aliquot parts in B equal to $\frac{1}{4}$ of A.

Answered by the proposer Mr. Mason.

$$A = 404 : B = 465.$$

Mr. Robert Fearnside's answer.

Let $2y$ = sum of the aliquot parts in A ; and $3z$ = sum of the aliquot parts in B : Then $3y$ will be = second number B ; and $4z$=A. Now $1 + 2 + 4 + z + 2z = 2y$; and $1 + 3 + y = 3z = 2y — 7$. Consequently the first number A will be found to be 20, and B = 33.

For $1 + 2 + 4 + 5 + 10$ is = $\frac{2}{3}$B ; and $1 + 3 + 11$ is = $\frac{1}{4}$A.

Mr. Grimmett's answer.

It is supposed that $\frac{1}{4}$ of A is an integer, because the sum of the aliquot parts of B is = to it ; and likewise $\frac{2}{3}$ of B is an integer for the same reason ; therefore the number A must be a multiple of 4 ; and the number B a multiple of 3. Make therefore $A = 4a$; its aliquot parts are $2a$, a, 4, 2, 1 the sum of which is $3a+7$. Put also $B=3e$; the sum of the aliquot parts of which is $e + 4$; and per question $3a + 7 = 2e$ $(= \frac{2}{3}$B$)$; also $e + 4 = 3a$ $(= \frac{1}{4}$A$)$. Whence a is found = 5, and $e = 11$. Consequently $4a = 20 =$ the number A ; and $3e = 33 =$ B.

THE PRIZE QUESTION, *by Mr.* Pointon.

A father at his death left his estate to two sons whose shares are to be determined from the following equations. Let the share of the eldest be denoted by b and that of the youngest by c. Let $b + c = a$; then $c + \dfrac{a}{5} = b$ by the father's will. A third person whom we shall call D hath a yearly income or annuity, that will fall to him seven years hence, and thence continue seven years, which being sold at 6 per cent. $= \sqrt{ba}$, and the rent of it $= \sqrt[3]{ca}$. Required the value of the estate?

Answered by Mr. Robert Fearnside.

Let $x^6 = a$, $y^6 = b$, $z^6 = c$; then $z^6 + \dfrac{x^6}{5} = x^6 - z^6$; consequently $x = z\sqrt[6]{\dfrac{10}{4}}$ and $x = y\sqrt[6]{\dfrac{10}{6}}$: Now by Ward's theorem, p. 271, we shall have (putting $r = \sqrt[3]{\tfrac{2}{3}m} = 1\cdot50363$, $n = 1\cdot06$) $\dfrac{nm^2x^3y^3 - m^2x^3y^3}{m - 1} = zzxx =$ rent; whence by substitution, &c. $x^2 = -\dfrac{m-1}{n-1} \times \dfrac{r}{sm^3} = 3\cdot5316$, then $x^6 = a = 44\cdot047579$, $b = 26\cdot423547$, and $c = 17\cdot61903$.

The same otherwise answered.

By the question $b - \dfrac{a}{5} = c$ and $b + c = a$, $\therefore 2b - \dfrac{a}{5} = a$, or $10b = 6a$, $\therefore a = \dfrac{10b}{6}$ and $c = \dfrac{2b}{3}$. Consequently ab, the square of the purchase money, is $= \dfrac{10bb}{6}$, and ca, the cube of the yearly rent, $= \dfrac{10bb}{9}$. Put $m^2 = \dfrac{10}{6}$, then $mmbb = \dfrac{10bb}{6}$ $\therefore mb =$ the purchase money, which (per Ward, p. 246, theo. 1.) will be $rtmb + mb$; and by the 2d theorem of Ward, p. 249, we have $\dfrac{2rtmb + 2mb}{ttr - tr + 2t} = u$. Put $2rtm = x$, $2m = z$, and $ttr - tr + 2t = d$; then $\dfrac{xb + zb}{d} = u$; therefore by the question $\dfrac{xb + zb}{d}$

o 2

$$= \sqrt[3]{\frac{10bb}{9}}, \text{ or } \frac{x^3b^3 + 3xxzb^3 + 3xzzb^3 + z^3b^3}{d^3} = \frac{10bb}{9}: \text{ hence } b =$$

$$\frac{1\frac{1}{9}ddd}{x^3 + 3xxz + 3xzz + z^3} = 101\cdot730562; \quad c = 67\cdot820375; \text{ the value}$$

of the estate $169\cdot550937$; the annuity or income $= 22\cdot577929$.

Additional Solution.

In this question for determining the three quantities, x, y, z, there are given the two equations $x + y = z$, and $y + \dfrac{z}{5} = x$:
And the third equation is to be made out from these following conditions, viz. that $\sqrt{xz}$ is the present value of a yearly rent which is expressed by $\sqrt[3]{yz}$, to continue for 7 years, but the first payment not to become due till the end of the 8th year; the interest of money being reckoned at 6 per cent.

From the two first equations we easily find $x = \dfrac{3z}{5}$, and $y = \dfrac{2z}{5}$; consequently the yearly rent becomes $\sqrt[3]{\dfrac{2z^2}{5}}$, and the present value

of it $\sqrt{\dfrac{3z^2}{5}}$ or $z\sqrt{\dfrac{3}{5}}$. But by any calculated table of reversions at 6 per cent. it appears that the present worth of one pound due 8 years hence is $\cdot6214$, the present value due 9 years hence $\cdot5919$, the same for 10 years $\cdot5584$, for 11 years $\cdot5268$, for 12 years $\cdot4970$, for 13 years 4688, and for 14 years $\cdot4423$; the sum of these is $3\cdot7126$ which is the present value of 1 for those 7 years, and consequently

$3\cdot7126 \sqrt[3]{\dfrac{2z^2}{5}} =$ the present value of $\sqrt[3]{\dfrac{2z^2}{5}}$ annuity for the same time,

and which must be equal to $z\sqrt{\dfrac{3}{5}}$, that is $3\cdot7126 \sqrt[3]{\dfrac{2z^2}{5}} = z\sqrt{\dfrac{3}{5}}$;

hence then z is $= 3\cdot7126^3 \times \dfrac{2}{3}\sqrt{\dfrac{5}{3}} = 3\cdot7126^3 \times \dfrac{2\sqrt{15}}{9} = 44\cdot042$.

Consequently $x = \dfrac{3z}{5} = 26\cdot4252$, and $y = \dfrac{2z}{5} = 17\cdot6168$.   H.

Another Additional Solution.

By the question $a = b + c$, $c + \dfrac{a}{5} = c + \dfrac{b + c}{5} = b$,

whence $b = \dfrac{3c}{2}$, $a = b + c = \dfrac{5c}{2}$, $c = \dfrac{2a}{5}$, $b = \dfrac{3c}{2} = \dfrac{3a}{5}$, $\sqrt{(ab)} =$

$= a\sqrt{\dfrac{3}{5}}$. Let p denote D's annuity or yearly income, then by the

question $p = \sqrt[3]{(ca)} = \sqrt[3]{\dfrac{2aa}{5}}$, and by Art. 197, page 196, vol. 2,

Dalby's Course $\left(\dfrac{1}{r^n} - \dfrac{1}{r^t}\right) \times \dfrac{p}{r-1} = \left(\dfrac{1}{r^n} - \dfrac{1}{r^t}\right) \times \dfrac{\sqrt[3]{\frac{2aa}{5}}}{r-1}$

expresses the present worth of the annuity p, to commence at the end of n years, and to continue till the end of t years.

Put $\left(\dfrac{1}{r^n} - \dfrac{1}{r^t}\right) \times \dfrac{1}{r-1} = e$ and then by the question we shall

have $e \sqrt[3]{\dfrac{2aa}{5}} = a\sqrt{\dfrac{3}{5}}$, $e^3 \times \dfrac{2aa}{5} = \dfrac{3a^3}{5}\sqrt{\dfrac{3}{5}}$, $a = \dfrac{2e^3}{3} \div \sqrt{\dfrac{3}{5}}$
$= \dfrac{10e^3}{9}\sqrt{\dfrac{3}{5}}$.

Take $t = 14$, $n = 7$ and $r = 1\cdot06$, then $e = 3\cdot71262$, $a = 44\cdot04279$, $b = 26\cdot42567$, $c = 17\cdot61711$.

It will be perceived that $e = \left(\dfrac{1}{r^n} - \dfrac{1}{r^t}\right) \times \dfrac{1}{r-1}$ is the pre-

sent worth of an annuity of 1l. to commence at the end of n years, and to continue till the end of t years.   L.

Questions proposed in 1732, and answered in 1733.

I. QUESTION 163, by Mr. Sam. Ashby.

There is a park in the form of a right-angled triangle, having a round walk touching the sides : there are also four straight paths terminating in the round walk, forming therein a trapezium ABCD; the side AB = 152·99 chains, BC = 191·11, CD = 93·03, and DA = 41·56 : Also the area of the park = 55296 square chains. It is required to determine the lengths of the three sides.

Answered by Mr. W. Grimmett,

Draw the diagonals BD and CA; and from the centre of the in-

scribed circle draw PM, RM, and NM:
Let AH be perpendicular to BD, as also
MO's to PN, PR, and RN,

Put $s =$ BC $= 191.11$, $c =$ AD $=$
41.56, $b =$ CD $= 93.03$, $d =$ BA $=$
152.99, and $a =$ BE. Then from
the similarity of triangles we have, as

$$d : a :: b : \frac{ab}{d} = \text{CE};$$ and after the

same way of reasoning DE will be had

$= \frac{bca}{sd}$; and AE $= \frac{ca}{s}$. But $a + \frac{bca}{sd}$

$=$ BD, and $\frac{ab}{d} + \frac{ca}{s} =$ AC; and then we shall have $\frac{baa}{d} + \frac{caa}{s} +$

$\frac{bbcaa}{sdd} + \frac{bccaa}{ssd} = cs + db$; which reduced will be $a =$

$$\sqrt{\frac{ssscdd + d^3bss}{bdss + ddcs + bbcs + ccdb}} = 167.65 = \text{BE};$$ and then BD will

be found $= 189.81$.

Again, from the circle and its inscribed triangle BAD, in which
the perpendicular AH is let fall, it will be as AH : DA :: BA : the
diameter of the circle $= 191.6832$; and consequently MO the radius
is 95.8416.

Put $b =$ area of the triangle PNR; $n =$ MO; $x =$ RN. Then

PN $= \frac{2b}{x}$, and $\sqrt{\left(xx + \frac{4bb}{xx}\right)} =$ PR; whence $nx + \frac{2bn}{x} +$

$n\sqrt{\left(xx + \frac{4bb}{xx}\right)} = 2b$: Per reduction there comes forth $xx - \frac{bx}{n}$

$- nx + 2b = 0$; which solved, x will be found $= 285.7084 =$ NR:
Then PN $= 387.079$, and PR $= 481.1031$ chains.   *Q. E. I.*

Additional Solutions.   1st. Algebraically.

Put a, b, c and d for the sides of the trapezium, and $\varphi =$ the angle
included between the sides a and b; then, because the figure is in-
scribed in a circle, the angle included between the sides c and d will
be $= 180^\circ - \varphi$; and by trigonometry, $a^2 + b^2 - 2ab \cos \varphi =$ the
square of the diagonal opposite to φ, which is also $= $ to $c^2 + d^2 -$
$2cd \cos (180^\circ - \varphi) = c^2 + d^2 + 2cd \cos \varphi$;

Whence $\cos \varphi = \dfrac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$.

AC $= \sqrt{(a^2 + b^2 - 2ab \cos \varphi)}$:
and the diameter $=$ AC $\times$ secant φ.   F.
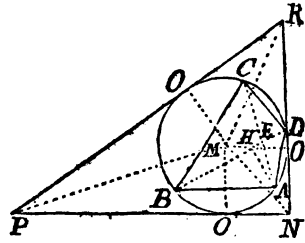
2d. ·Geometrically.

· In this question it is required to inscribe a quadrilateral whose sides are given in a circle, which may be done as follows: Suppose ABCD to be the quadrilateral required; join AC, BD and draw DF parallel to AC, meeting BC produced in F; then because the angles ABE, DCE are equal; and also those at E, the triangles ABE, DCE are similar; therefore AB : DC :: BE : CE (by the parallels) :: BD : DF; therefore BD has to DF a given ratio. Again, because the angle BCF is = DAB, and the angle CDF = ACD = ABD, the triangles ABD, DCF are similar, and therefore AB : AD :: DC : CF: Wherefore CF is a given line, and consequently BF is a given line; and since BD has to DF a given ratio, the locus of D is a given circle (Apoll. Loci Plani ii. 2.), and CD being also a given line, we have, this construction.
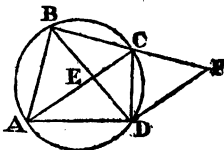
On BC produced take CF a fourth proportional to AB, AD and DC, and describe the circle which is the locus of D, so that any two lines being inflected to it from B and F they may have the given ratio of AB to DC; then from the centre C, with the distance CD, describe an arc to intersect the other circle in D, lastly from the centres D and B, with distances equal to the given sides DA, BA, describe arcs to intersect in A; join DA, BA, and the quadrilateral will be constructed. J. L.

II. QUESTION 164, by Mr. John Turner.

Two men, A and B, buy a piece of ground in an unknown northern latitude: but it was observed that on a certain day in the year, also unknown, the sun's altitude upon the south part of the meridian, at the said place, was 42° 30′; and upon the north part of the meridian, his altitude above the horizon was 4° 30′: The limits of the ground were to be marked out by the shadow of the vertex of a tree 20 yards high, on that same day when the altitude of the sun on each part of the meadow was observed as above-mentioned. It is required hence to find the latitude of the place, and the sun's declination; and also the share of the ground belonging to each man; A being to have for his part the greatest triangle that can be cut out of the said conic section described by the shadow of the tree's top, and B to have the remainder.

Answered by Mr. Rob. Fearnside. ·

It is obvious that the declination of the sun is equal to half the sum

of the meridian altitudes, which consequently is 23° 30′; and the latitude 71°.

Now the tree being supposed to be placed in H, it is evident, as the sun does not set, that its summit will describe the ellipsis AGFDBDA; therefore, by plain trigonometry, AH will be found = 21·826, and BH = 254·124; and (finding the altitude of the sun when due east or west) GH = 43, and consequently FC the semi-conjugate diameter = 79·5.
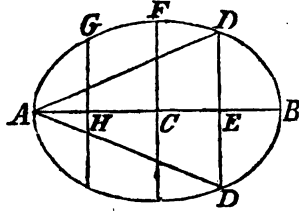
Then put $AB = 2a$, $FC = b$, $CE = x$, and $DE = y$. Then, per conics, $aa - xx : yy :: aa : bb$: therefore $y = \dfrac{b}{a} \sqrt{(aa - xx)}$. Now $(a + x) \times \dfrac{b}{a} \sqrt{(aa - xx)}$, = the area of the triangle DAD, must be a maximum; which put into fluxions and ordered, x will be $= \frac{1}{2}a$, and the area of the greatest triangle will be = 7124·37 = 2 a. 3 r. 32 p. = A's share.

and 27335·88 = 4   0   29   = B's share.

Mr. *Grimmett* having discovered a new property of the ellipsis, after a solution to this question, concludes with this other following method:

Supposing a circle inscribed in the ellipsis, then it will be as the radius of the inscribed circle, is to the perpendicular height of the equilateral triangle inscribed therein; so is the semi-transverse of the ellipsis, to the perpendicular of the greatest triangle inscribed in the ellipsis = 206·9652.

The answer will be 14277·7 = 2 a. 3 r. 32 p. = A's share.

20251·5 = 4   0   29   = B's share.

Remark. Having found the declination and latitude as in the first solution above, viz. by taking half the sum of the greatest and least altitudes for the declination, and then by taking the complement of the difference between this declination and the greatest altitude for the latitude of the place, which are general rules; next compute the altitude when due east or west, and then say as radius: the height of the tree :: the cotangent of each of these three altitudes : each of the three lines HA, HB, HG.——All the rest of the first solution above is very clear.

The truth of Mr. Grimmett's theorem above may appear thus: From Mr. Fearnside's solution we find that the altitude of the greatest triangle is 3-4ths of the transverse axe; and by geometry we know that the altitude of an equilateral triangle is also 3-4ths of the diameter of its circumscribed circle; wherefore as the diameter of any circle is to the altitude of its inscribed equilateral triangle, so is the transverse axe of an ellipse to the altitude of its greatest inscribed triangle.——We may hence remark also that the equilateral is the greatest triangle that can be inscribed in a circle.   H.

III. QUESTION 165, *by Mr.* Tho. Grant.

There is a cask in the form of the frustum of a spheroid; the length is 45 inches; the bung diameter is 36 inches; greater head diameter 32, and less head diameter 27 inches. Required a general theorem expressing the content of the cask; and find what part of the length lies on each side of the bung diameter.

Answered, by Mr. R. Lovatt.

Given $\begin{cases} \text{CF} = & 45 = d, \\ \text{NH} = & 18 = b, \\ \text{BC} = & 16 = n, \\ \text{EF} = & 13{\cdot}5 = m. \end{cases}$
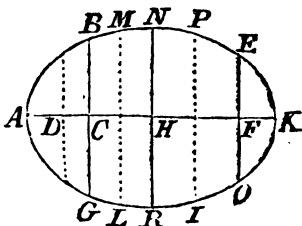
Put $a = \text{AH}$, $e = \text{CH}$, and $h = \dfrac{bb}{dd}$.

Then, per conics, $aa : bb :: aa -$
$ee : nn$; hence $\dfrac{aabb - aann}{bb} = ee$.

Substitute $pp = \dfrac{bb - nn}{bb}$: Then $ap = e$; and hence $d - ap = \text{HF}$.

Again, $aa : bb :: aa - dd + 2dpa - ppaa : mm$; and hence $aamm = aabb - ddbb + 2dpbba - bbppaa$. Substitute $c = bb - mm - ppbb$; and $k = 2dpbb$; then $a = \sqrt{\left(\dfrac{ddbb}{c} + \dfrac{\frac{1}{4}kk}{cc}\right)} - \dfrac{\frac{1}{2}k}{c} = 40$.

Hence $\text{CH} = 18{\cdot}311$, and $\text{HF} = 26{\cdot}68$.

To find the content; put $m = \text{AH} = 40$, and $n = \text{CH} = 18{\cdot}311$; find 18311000000000000000 ordinates rightly applied betwixt c and H, which will give the equal cylinder; thus put $d = 1831100$, &c. then $\dfrac{*\,3mmd - \frac{1}{2}nn\frac{1}{2}n - nnd}{3d} \times \frac{1}{4}h =$ the square of the equal cylinder; and $\dfrac{*\,\sqrt{(3mmd\frac{1}{2}nn - \frac{1}{2}nn - \frac{1}{2}n - nnd)}}{3d} \times 4h = 34{\cdot}7198814$

$= \text{ML} =$ the diameter of the cylinder $= \text{BNRG}$; and by the same method the cylinder $=$ to NEOR is $= 33{\cdot}2216007 = \text{PI}$; hence the content is $\begin{cases} \text{BNRG} = & 61{\cdot}470027 \\ \text{NEOR} = & 82{\cdot}041786 \end{cases} 143{\cdot}5118$ ale gallons.

The same answered by Mr. J. Turner.

Put $\begin{cases} \text{bung diam.} & 36 = m \\ \text{greater head} & 32 = n \\ \text{lesser head} & 27 = s \\ \text{length} & 45 = 2t \end{cases}$ and AC, the part wanting, $= 2x$. Then is $\text{HF} = t + x$, and $\text{CH} = t - x$.

As $tt + 2tx + xx : mm - ss :: tt - 2tx + xx : mm - nn$, by the

property of the ellipsis, (as per Ward's Introduct. p. 448.) therefore $ttmm + 2tmmx + 2mmxx — ttnn — 2tnnx — nnxx = ttmm — 2tmmx + mmxx—ttss+2tssx—ssxx$; by reduction and transposition, $ssxx—nnxx + 4mmtx — 2tnnx — 2tssx = ttnn —ttss$.

Put $ss — nn = — b$, and $4mmt — 2tnn — 2ltss = c$, and $ttnn — ttss = d$; then $— bxx + cx = d$; and, extracting the root, $x =$

$$\frac{c}{2b} — \sqrt{\left(\frac{cc}{4bb} — \frac{d}{b}\right)} = 4\cdot10 \text{ inches}; \text{ then } 2x = 8\cdot20 \text{ inches} =$$

the part wanting DC : from whence we find HF the distance from the greatest bulk of the cask to the lesser head to be $= 26\cdot6$ inches, and CH to the greater head $18\cdot4$ inches. And the content is $143\cdot9$ ale gallons.

Mr. *Grimmett*, after two different solutions to this question, delivers this theorem :

THEOREM. The square root of each difference between the square of half the bung diameter and the square of half the diameter of each head,.put into one sum : It will be as the sum is to either of those roots, so is the length of the cask to the distance of the respective head from the bung.

Remark. By the nature of the ellipse, $\sqrt{(HN^2—CB^2)} : \sqrt{(HN^2—FE^2)}$ :: CH : HF ; and, by composition, &c. $\sqrt{(HN^2—CB^2)} + \sqrt{(HN^2— FE^2)}$: CF :: $\left\{ \begin{array}{l} \sqrt{(HN^2 — CB^2)} : CH \\ \sqrt{(HN^2 — FE^2)} : HF \end{array} \right\}$, which therefore both become known; and this is Mr. Grimmett's theorem, mentioned above.

Again, by the nature of the ellipse, $\sqrt{(HN^2 — CB^2)} : CH$ :: HN : HA the semi-transverse : which being thus found, the contents of the two parts BNRG, NEOR, of the cask being computed separately by the common rules, their sum will be the whole content.——Or indeed their contents are easily computed without the transverse axe by rule 1, page 278, of my Mensuration.

N. B. The two expressions marked * in Mr. Lovatt's solution are wrong printed ; but they are here given as they stand in the original, as it is not easy to distinguish what are the true expressions meant. H.

IV. QUESTION 166, *by Mr.* Christ. Mason.

Required a theorem or formula expressing all perfect numbers ; and give an example in finding all such numbers from 1 to 10000000.

Answered by Mr. Grimmett.

If from unity be taken how many numbers soever in double proportion continually, until the whole added together be a prime number ; and if this whole be multiplied by the last term of the series which constitutes the prime, the product will be a perfect number. 36 Euclid 9.

From such a series it may be observed, that any term made less

by unity, will be $=$ the sum of all the preceding terms. Put there-fore $a = 2$; and $x =$ its variable exponent (for in the first opera-tion it will represent 1, in the next 2, and then 3, &c. till it be raised to a^{x+1} and being lessened by unity may be a prime number. Thus

$$x \text{ being } \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \&c. \end{Bmatrix} (a^{x+1} - 1) \times a^x = a^{2x+1} - a^x = \begin{Bmatrix} \left. \begin{matrix} 6 \\ 28 \end{matrix} \right\} \text{perf. num.} \\ - \text{ abounding,} \\ 496 \text{ perf. num.} \\ - \text{ abounding,} \\ 8128 \text{ perf. num.} \\ \&c. \end{Bmatrix}$$

Whence the perfect numbers are 6, 28, 496, and 8128, all the per-fect numbers required per question.

<center>*Mr.* C. Mason, *the proposer, gives this rule:*</center>

$(1 + 2) \times 2 = 6$; $(1 + 2 + 4) \times 4 = 28$; $(1 + 2 + 4 + 8)$ $\times 8 = 120$, &c.

<center>*Mr.* Sam. Ashby *answers thus:*</center>

The canon. If from any power of 2 be subtracted unity, and that remainder be a prime number, multiply it by half the said power, and that product will be a perfect number.

<center>*Mr.* Robert Fearnside's *answer.*</center>

Let $y^n x$ be the number sought; its aliquot parts will be $1 + y + y^2 + y^3$, &c. till the exponent becomes n; and $x + yx + y^2x + y^3x$, &c. till the number of terms be likewise n; then, from the nature of a perfect number, $1 + y + y^2 + y^3 +$ &c. $+ x + yx + y^2x + y^3x +$ &c. $= y^n x$; and consequently $x = \dfrac{1 + y + y^2 + y^3 \&c.}{y^n - 1 - y - y^2 - y^3}$: Now, that x may be

a whole number, it is requisite that $y^n - 1 - y -$ &c. be $= 1$, which only happens when y is $= 2$; whence the canon required becomes $2^n x$. If $n = 1$, x will be $= 1 + 2$, and the first perfect number $= 6$. If $n = 2$, x will be $= 1 + 2 + 4$, and the second perfect number will be $= 28$. If $n = 4$, the third perfect number $= 496$. If $n = 6$, the fourth perfect number is 8128. Which are all the perfect numbers from unity to ten millions.

Woolfius, in his Elem. Math. supposes n to be successively the numbers 1, 2, 3, 4, 5, &c. which will not hold when n is supposed $=$ the odd numbers 3, 5, 7, 9, &c. and Mr. *Cunn's* rule for finding a perfect number will not find all the above numbers: See p. 12 of his Decimals.

Additional Solution.

By *Eucl.* IX. 36, : $1 + 2 + 2^2 + 2^3 + 2^4 +$ &c. to 2^n : $\times$ 2^n is a perfect number when the sum of the series is a prime number ; but the sum of the geometrical series is $2^{n+1} - 1$, therefore $(2^{n+1}-1) \times$ 2^n is a perfect number when $2^{n+1} - 1$ is a prime number. Taking $n = 0$; then $2^{n+1} - 1$ is $= 1$ a prime, and $1 \times 2^0 = 1 \times 1 = 1$ the first perfect number : If $n = 1$; then $2^{n+1} - 1 = 3$ a prime, and $3 \times 2^1 = 6$ the next perfect number : If $n = 2$; then $2_{n+1} - 1 = 7$ a prime, and $7 \cdot \times 2^2 = 28$ the 3d perfect number : If $n = 3$; then $2^{n+1} - 1 = 15$ which is not a prime, and therefore $15 \times 2^3 = 120$ is not a perfect number : In like manner it will appear that no other greater odd number can be put for n so as to make the expression $(2^{n+1} - 1) \times 2^n$ a perfect number ; n must therefore be always an even number for finding the other perfect numbers ; but it cannot be any even number, as some have falsely asserted. Dr. Harris says that there are only ten perfect numbers between 1 and 1,000,000,000,000.

This rule of Euclid's only demonstrates that a number found by it will be a perfect number ; but neither it nor any other that I know of, shews that there may not be other perfect numbers besides those found by this rule. H.

V. QUESTION 167, *by Mr.* Turner.

Let there be a triangle whose three sides are given, viz. 415, 353, and 488 : And upon the three angular points, as centres, let there be described three circles whose radii are 130, 80, and 70 : Let a fourth circle be drawn, which shall touch these three circles. It is required to find its diameter ?

Answered by Mr. Turner, *the proposer.*

This problem is taken from lemma 6, book 1, of Sir Isaac Newton's Principia, where the geometrical construction may be seen.

Having the three sides of the triangle, the perpendicular Fn, and segments Dn and En, may be found.

Draw cs perpendicular to DE, and cm perpendicular to Fn; and put
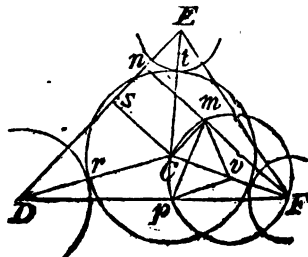
$cr = ct = cv = a$, $\text{Fn} = 345 \cdot 8 = b$,
$cm = sn\quad = e$, $\text{En} = 70 \cdot 7 = c$,
$mn = cs\quad = u$, $\text{D}n = 344 \cdot 3 = d$.   Then is
$\text{D}r = 130\quad = r$, $\text{D}s = d - e$, and
$\text{F}v = 70\quad = s$, $\text{E}s = c + e$; also
$\text{E}t = 80\quad = n$, $\text{F}m = b - u$: Then

(1) $rr + 2ra + aa = dd - 2de + ee + uu$
(2) $nn + 2na + aa = cc + 2ce + ee + uu$ } per 47 Euc. 1.
(3) $ss + 2sa + aa = bb - 2bu + uu + ee$

Subtract the third step from the first, and transpose the terms, and dividing by $2b$, we shall have

(4) $u = \dfrac{rr - ss + 2ra - 2sa + bb - dd + 2de}{2b}$.

Put $rr - ss + bb - dd = f$; and $2r - 2s = g$; then $u = \dfrac{f + ga + 2de}{2b}$; and $uu = \dfrac{ff + ggaa + 2fga + 4fde + 4dgae + 4ddee}{4bb}$.

Subtract the second step from the first, and transposing makes
$2de + 2ce = dd - cc - rr + nn + 2nu - 2ra$.

Put $2d + 2c = h$; $dd - cc - rr + nn = k$; and $2n - 2r = -l$;

(5) Then $e = \dfrac{k - la}{h}$; and $ee = \dfrac{kk - 2kla + llaa}{hh}$.

In the first step substitute the value of uu found, then
$rr + 2ra + aa = dd - 2de + ee +$
$\dfrac{ff + 2fga + ggaa + 4fde + 4dgae + 4ddee}{4bb}$, or $4bbrr + 8bbra +$

$4bbaa - ggaa - 2fga - ff - 4bbdd - 4bbee = 4ddee + 4fde - 8bbde + 4dgae$: Put $4bb + 4dd = m$; $4fd - 8bbd = -n$; and $4dg = p$; then $4bbrr + 8bbra + 4bbaa - ggaa - 2fga - ff - 4bbdd = mee - ne + pae$; for e and ee put the values above found and bring it out of the fractions, will give this quadratic equation :

$$
\left.\begin{array}{l} + 4bbhh \\ + plh \\ - gghh \\ - mll \end{array}\right\} aa - \left.\begin{array}{l} + 8bbhhr \\ + 2mkl \\ 2fgh \\ - nlh \\ - pkh \end{array}\right\} a = \left\{\begin{array}{l} 4bbddhh \\ + ffhh \\ + mkk \\ - 4bbrrhh \\ - nkk. \end{array}\right.
$$

In numbers $aa + 195 \cdot 085\, a = 52778 \cdot 24$ when divided by the co-efficients of the highest power; and, extracting the root, $a = 152 \cdot 048$ the radius of the circle touching the three circles given in position and magnitude; consequently the diameter is $= 304 \cdot 096$ the answer.

Mr. *George Brown*, after giving the true answer in a very concise way, says, since it is not limited how the three circles shall be placed on the angular points, it will admit of so many answers as the circles are to be varied.

Mr. Rich. Lovatt's *answer to the same.*

Put $b = \dfrac{488^2 - 130^2 - 70^2}{488 \times 2}$; $d = \dfrac{130 - 70}{488}$; $n =$

$\dfrac{353^2 - (80 - 70)^2}{353 \times 2}$; $m = \dfrac{80 - 70}{353}$; $p = \dfrac{\text{DFE}}{2}$; and $a = \text{CF}$.

Then, per axiom 4, $b - ad = p\text{E}$; and $n - am = m\text{F}$: and CF will be the diameter of a circle, whose periphery will pass through *mcr*; and (20 Eucl. 3.) the angle at the centre, v, is double to the angle, F, at the periphery; also (5 Eucl. 1.) the angle $vpm = vmp$. Put $h = $ s. $\angle vmp = 33° 34'$; $k = $ s. $\angle mvp = 112° 56'$; then h :

$\dfrac{a}{2} :: k : \dfrac{ak}{2h} = mp$. And by that noble theorem which I mentioned in the Diary of 1731, quest. 143, we have this analogy : $(b - ad) \times$

$(n - am) : \dfrac{b - ad - n + am}{2} + \dfrac{ak}{2h} \times \dfrac{- b + ad + n - am}{2}$

$+ \dfrac{ak}{2h} :: 1 : pp$; and $pp \times (b - ad) \times (n - am)$

$= \dfrac{b - ad - n + am}{2} + \dfrac{ak}{2h} \times \dfrac{- b + ad + n - am}{2} + \dfrac{ak}{2h}$.

In numbers, $1682aa + 9·071a = 9473·58 + 1023·9$, and $a = \sqrt{63140·3} - 27 = 224·273$; hence $224·273 - 70 = 154·273 =$ *cr*, which doubled is the diameter $308·56$ of the circle required.

Remark. This is one of the problems of Apollonius on Tangencies, and is constructed by his restorers Vieta, and our countryman the Rev. Mr. John Lawson, who has lately published an English restoration of this piece of Apollonius's works, where it appears that the problem hath several cases, according as the fourth circle is to touch the other three, either all internally or all externally, or else some internally and the rest externally.

This problem has also been attended to by several other respectable persons, it being constructed by Sir Isaac Newton, in lemma 16, lib. 1. of the Principia; and his Universal Arith. prob. 47 ; by the Marquis de l'Hospital in his Sectiones Conique, lib. 10, ex. 4, cor. 1; and by Mr. Tho. Simpson, at the end of his Geometry.

Concerning this problem I shall also insert the following extract from the Histoire des Mathematiques, par M. Montucla, tom. i. p. 263.

Vieta, in a dispute which he had with Adrianus Romanus, proposed to him this question. The solution which Romanus gave to it, though obvious, was very indifferent, viz. by determining the centre of the circle sought in the point of intersection of two hyperbolas ; for as the problem is a plane one, it may be solved by plane geometry ; by this Vieta solved it, and very elegantly: his solution is the same as that given in Newton's Universal Arithmetic. Another solution may also be seen in the 1st book of the Principia (this question being there necessary for some determinations in Physical Astronomy), wherein

Newton, by a remarkable dexterity, reduces the two solid loci of Romanus to the intersection of two right lines.—Moreover, Descartes attempted to solve this problem by the help of the Algebraical Analysis, but without success; for of the two solutions which he derived from thence, he himself acknowledges (see Lett. tom. iii. let. 80, 81.) that one furnished him with so complicated an expression, that he would not undertake to construct it in a month ; whilst the other, though somewhat less complicated, was not so very simple, as to encourage him to set about a construction of it.——Lastly, the Princess Elizabeth of Bohemia, who, it is well known, honoured Descartes with her correspondence, deigned to communicate a solution to this Philosopher ; but as it is deduced from the algebraical calculus, it labours under the same incoveniencies as that of Descartes. H.

VI. QUESTION 168, *by Mr.* Chr. Mason.

Three persons, A, B, C, each staked a guinea to play for at dice, the game to be 15 ; when growing tired they agreed to part the money according to the chance which each had of winning at the time : now A had got 10, B 8, and C 5 ; what part of the money ought each to have.

Answered by Mr. Rob. Fearnside.

Let A, B, C, represent the three players ; A wants 5 of being up, B 7, and C 10. Now it is plain the game will be ended in 20 throws at most; then A + B + C must be raised to the 20th power ; and as the players here are supposed equal, the co-efficients of every term where the 5th power of A and upwards, including the 20th, is found, are to be added together, as also the co-efficients where the 7th power of B and upwards is found, and the co-efficients of the 10th power of C and upwards are to be added also together ; these three totals will be in proportion to one another as the respective shares they are to have of the guineas.

	l.	s.	d.	
He who got 10 must have	2	3	10	+
He who got 8 must have	0	16	$1\frac{3}{4}$	+
He who got 5 must have	0	3	0	+

Mr. *John Ommanney*'s numbers are the same.

Remark. The method of solving questions of this kind, may be seen at page 43 or 192, of De Moivre, or in some other books on Chances. H.

THE PRIZE QUESTION, *by Mr.* Rob. Fearnside.

To determine the longest cylindrical pole, 1 foot diameter, that can be put up a chimney, whose width at the mantle tree is 4 feet,

and height of the mantle tree from the floor 8 feet. It is also re-
quired to determine the greatest conical pole that can be put up the
said chimney, the slant side of the cone being to the diameter of its
base as 4 to 1?

Answered.

Let FI represent the floor, and ADEM the chimney; then put
BC $= d$, ED $= b$, DI $= c$; LB $= x$. Now,

per similar triangles, $x : d :: d : \dfrac{dd}{x} =$ AB;
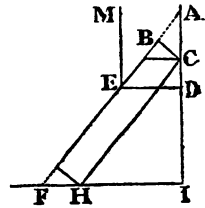
and $d : \dfrac{dd}{x} :: b : \dfrac{bd}{x} =$ AD. After the same

manner of reasoning we shall find AC $= \dfrac{d}{x}$

$\sqrt{(dd + xx)}$; CI $= \dfrac{bd + cx}{x} - \dfrac{d}{x} \sqrt{(dd + xx)}$;

and CH $= \dfrac{bd + cx}{dx} \sqrt{(dd + xx)} - \dfrac{dd}{x} - x$; which must be a

maximum; consequently put into fluxions, &c. the following equa-
tion will come out; *i. e.*
$(cc - dd) x^6 + d^4 x^4 - 2bcd^3 x^3 + d^6 x^2 + (bbd^6 - d^6) = 0.$

Brought into numbers and reduced $x = 9.74$ inches; and conse-
quently CH $= 175.2$ inches $= 14$ feet 7 inches 2 tenths, the length
of the cylindric pole.

For the Conical Pole.

Let ED $= b$; DI $= c$; and $r = 4$; CD $= x$; HO $= 2y$: then 1 :
$r :: 2y :$ HC $= 2ry$; GC $= y\sqrt{(4rr - 1)}$
$= ny$ (putting $\sqrt{(4rr - 1)} = n$); as $2ry$

$: ny :: 2y :$ PH $= \dfrac{ny}{r}$; PO $= \dfrac{y}{r}$. Again x
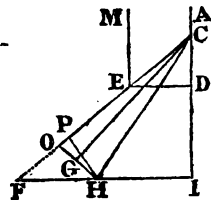
$: \sqrt{(bb + xx)} :: c + x :$ FC $= \dfrac{c + x}{x}$

$\sqrt{(bb + xx)}$; and $x : b :: \dfrac{ny}{r} :$ PF $= \dfrac{bny}{rx}$.

Consequently $\dfrac{bny}{rx} - \dfrac{y}{r} = \dfrac{c + x}{x} \sqrt{(bb + xx)} - 2yr$. Ergo (putting

$2rr - 1 = m) \; y = \dfrac{cr + rx}{bn + mx} \sqrt{(bb + xx)}$. Now it is evident

that GC $= \dfrac{cnr + nrx}{bn + mx} \sqrt{(bb + xx)}$ must be a maximum; which in

fluxions, &c. the following equation will come out, *viz.* $x^3 + \dfrac{2bn}{m} x^2$

$+ \dfrac{bcn}{m} x = b^2 c - \dfrac{b^3 n}{m}$; brought into numbers and reduced, gives x

$= 44\cdot 88$ inches, and GC $= 168\cdot 48$ inches $= 14$ feet 0 inches, 48, the length of the conical pole required.

Remark. It is true the process above will bring out the longest pole which can be put quite into or up the chimney, but some of the expressions used in it are very improper : thus the expression for CH in the former case, and CG in the latter, is not a maximum, but a minimum; for it has no maximum but infinity; and the thing to be found, though it be the longest pole that can be put quite up the chimney, is the minimum of CH or CG, that is the shortest pole which can rest with one end on FI, the other on AI, and its side touch the point E : for it is evident that whatever way this line be moved from this narrowest or shortest position, its side will fall below the point E, and so it may be put up the chimney ; but a longer cannot be put into the said shortest position, and therefore not up the chimney.

The former part of the process, for determining the length of the cylinder, when considered of no thickness, may be brought out by a simple cubic equation thus :

Put AD $= z$; also DE $= b$, and DI $= c$, as above. Then AI $= c$

$+ z$, and $z : b :: c + z : \dfrac{c+z}{z} \times b =$ IF; hence $\dfrac{c+z}{z} \sqrt{(bb+zz)}$

$=$ AF, which must be a minimum. This put into fluxions, &c. we get $z^3 - b^2 c = 0$; hence $z = \sqrt[3]{b^2 c} =$ AD. And therefore the length

of the rod or line AF in this case is $\dfrac{c + bq}{q} \sqrt{(1 + qq)}$, where $q =$

$\sqrt[3]{\dfrac{c}{b}}$.　H.

—●✕●—

Questions proposed in 1733, and answered in 1734.

I. QUESTION 169, *by Mr.* Will. Grimmett.

In a certain dictionary, under the word *Conoid*, it is said the so- lidity of an hyperbolic conoid is to its circumscribing cylinder as 3 to 10; and in an appendix of fluxions the same is also asserted; which is certainly false. Be pleased therefore to investigate the ex- pression that does expound their ratio ; and when you are in this

way of thinking, suppose the generating hyperbola to become the plane of a west declining dial, in the latitude of 50° north, and the focus to be the centre of the same, in which, if you erect a wire perpendicular to the plane, the sun on its first shining on the plane, the 12th of May, will cast the shadow of the wire, so erected, exactly on the hour line of 8. Quere, the declination of the plane?

Answered.

Let the abscissa of the hyperbola be $= x$, the ordinate $= y$, the parameter $= b$, and the transverse $= a$; then the nature of the curve will be expressed by $yy = bx + \dfrac{bxx}{a}$; then by the doctrine of fluxions $\dfrac{pbxx}{4r} + \dfrac{pbxxx}{6ra}$ will express the solidity of a conoid whose altitude is $= x$, and the radius of its base $= y$: And the solidity of its circumscribing cylinder will be expressed by $\dfrac{pbxx}{2r} + \dfrac{pbxxx}{2ra}$: Consequently their ratio will be expressed by $\frac{2}{3}a + x$ to $3a + 3x$, which is agreeable to that deduced from the method of indivisibles.

If we consider such a one whose altitude is equal to the transverse axis of the generating hyperbola; the ratio will be expounded by that of 5 to 12.

When the sun is in the plane of any dial, the shadow of the style, and that of a wire erected perpendicular to the plane in the centre of that dial, will be coincident.

Demonstration. Let ABCD represent the plane of a dial coincident with the plane of the paper; then the point E will be the perpendicular wire, EF the hour line of 12, EH the style, EG the substyle; ss a parallel the sun describes, which will here become an ellipsis. Now the sun in some point of the orbit will be in the plane of the dial, suppose at s or s; then its plane by the line drawn from the sun through the centre of the dial will give the shadow of the style on the plane, as SP or SP; for a line drawn from the sun through any other point of the style more remote from the plane will not fall on the plane. Again the shadow of a wire erected perpendicular on a plane, when the sun is in that plane will be coincident with a line drawn from the sun through the intersection of the wire with that plane (which is here said to be the centre of that dial) consequently when the sun is in the plane, the shadow of a style (however inclined) and the shadow of the wire erected perpendicular in the centre will be coincident. *Q. E. D.*
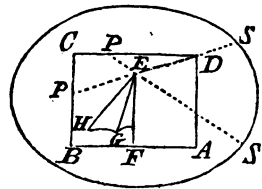
Therefore from the latitude of 50 N. the declination on the 12th of May and hour of 8, the azimuth will be found 99° 48′ from the north, which is the declination of the plane from the point.

II.   QUESTION 170, *by Mr.* Sam. Ashby.

If upon each leg AB and BC, including the right angle, be drawn a square BD and BE; and the lines DC and EA, which cut the said legs at F and G. I say, BF and BG are equal, and are each a mean proportional between the segments AF and CG; that is, as AF : FB :: FB : CG, &c. Quere, the demonstration geometrically?
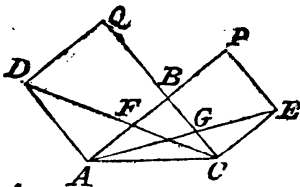
Answered by Mr. Rob. Fearnside.

It is plain, by similar triangles, that CB + AB : AB :: BC : FB. Again CB + AB : BC :: AB : GB. Permutando CB + AB : AB :: BC : GB. Ergo FB $=$ GB.

For the other part of the demonstration; by similar triangles AB ($=$ AD) : AF :: BC : BF; and AB : BG :: BC ($=$ CE) : CG. Therefore, *ex equo* AF : BG :: BF : GC. *Q. E. D.*

A Grubean Lady's answer to the same.

The triangles APE and ABG, being similar, it is AP : PE :: AB : BG, (by Euclid 4. VI.); that is AB + BC : BC :: AB : BG, because all the sides of a square are equal. In like manner the triangles CQD and CBF, being similar, it is QC : CB :: DQ : FB; *i. e.* AB + BC : BC :: AB : FB. Therefore FB is $=$ BG. Again, the triangles DAF and FBC, and the triangles ABG and ECG are similar; therefore (DA or) AB : BC :: AF : FB; and AB : BC ($=$ CE) :: BG : GC. Whence AF : FB :: BG (or BF) : GC (by Euc. 11. V.) *Q. E. D.*

III.   QUESTION 171, *by Mr.* Geo. Brown.

Three ships departed from a certain port in latitude 51° N. The first sailed S.E. 33·8, the second S.S.E. 49, and the third S.S.W. 35·5 leagues, then each vessel changed her course, and after sailing an equal distance they all met again. Required the second course and distance of each of the ships, with the latitude of the place arrived at. Required also the course and distance from the port first sailed from?

Answered by Mr. J. Turner.

This is altogether solved by trigonometry: For, first, in the tri-
angles BAS, CAS, there is given two sides and
a contained angle, to find BS, and the ∠BSA,
and CS and the ∠CSA; add these two angles
together, and then you have in the △BSC
two sides and a contained angle, to find BC:
Now in the △Bnc, the angle at n is double
the ∠BSC, and being isosceles, it is easy to
find Bn = Cn = Sn = 19·44 leagues, each
ship's 2d distance run, from B, S, C, to meet
at n. Again, in the △ABC, find the ∠s ABC
and ACB, to which add the ∠s CBn, BCn, we
find that the 2d course of the first ship steered
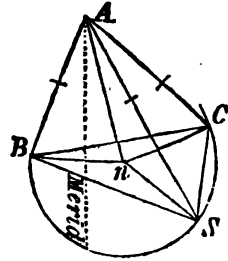
S.S.W. from B to n is E. by N. 5° 26' easterly,
2nd ship S.E.   c to n is W.S.W. 1 37 westerly,
3rd ship S.S.E.   s to n is N.W.   3 12 northerly.

Lastly, in the △BAn, there is given BA, Bn, and the angle con-
tained; to find An = 31·4 leagues, the distance from the port A to
where they all meet: And by the ∠BAn found, the course from A to
n is S. by E. 34' southerly. The difference of latitude of the ships
when at n, = 30·7 leagues.

IV: QUESTION 172, *by Mr.* Chr. Hale.

Let the rectangle of two right lines be 2332800 square inches;
then the length of a third right line, which together with the other
two shall include the greatest area possible, will be equal to the
height of All Saints Steeple, Derby: Required the steeple's height?

Answered.

I have received a great many answers to this question, but they
generally agree that it is unlimited, or else they do not rightly under-
stand the proposer's meaning; but if that product be broke into two
equal lines, making the legs of a right-angled triangle, the hypo-
thenuse will be 180 feet. I shall here give you the proposer's own
solution.

Put $dd = 2332800$, and $a = $ AC. $a : d + d :: d - d : 0 = $
AD — DC: And $\frac{a}{2} = $ AD.

$\frac{4ddaa - aaaa}{16} = $ greatest area squared.
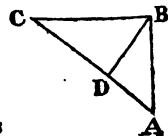
In fluxions, $8ddaa - 4a^3a = 0$: Reduced gives
$a = 2160$ inches. The exact height of All Saints
tower 60 yards.

Additional Solution.

Since the product of any two sides of a triangle drawn into the sine of their included angle produces double the area; therefore when the product of the sides is given, the area will be as the sine of the angle; but the right angle has the greatest sine, therefore the triangle is right-angled when a maximum. And then into whatever two parts the given product is broken, the area will be still the same; but then the hypothenuse will vary, and is shortest when the triangle is isosceles, in which case it is $\pm \sqrt{2} \times 2332800 = 2160$ inches $= 60$ yards. H.

v. QUESTION 173, by Mr. John Turner.

Two ships sail from a certain port to two other ports in the same latitude, viz. 35° 34′ N. The first ship sails between the south and east, and the other between the south and west, the ratio of the distances run are as 12 to 5, and the angle formed by the two courses 112° 29′; the departure of the ship which went to the westward is 229·7 miles. Required the latitude of the port sailed from and the course and distance run by each ship?

Answered by Mr. Fearnside.

Put $b = \text{CD}$, $d = \frac{12}{5}$, sine of the sum of the courses 112° 29′ $= c$, cosine $= a$, $x = \text{AE}$.

Then $a : x :: 1 : \text{AD} = \dfrac{x}{a}$ and $\text{DE} =$

$\dfrac{cx}{a}$, $\text{BA} = \dfrac{dx}{a}$, $\text{CA} = \sqrt{\left(\dfrac{xx}{aa} - bb\right)}$. Then

$\text{BD}^2 = xx + \dfrac{2dxx}{a} + \dfrac{ddxx}{aa} + \dfrac{ccxx}{aa} = $ (put-

ting $p^2 = 1 + \dfrac{2d}{a} + \dfrac{dd}{aa} + \dfrac{cc}{aa}$) $ppxx$, therefore $px = \text{BD}$, and

$px - b = \text{BC}$: Per similar triangles, $px : x + \dfrac{dx}{a} :: \dfrac{dx}{a} : px - b$;

hence x will be found $= \dfrac{pbaa}{ppaa - ad - dd} = 134 \cdot 28$ miles. Consequently $\text{BA} = 842 \cdot 6$ miles, $\text{AD} = 351 \cdot 15$ miles, $\text{BC} = 799 \cdot 79$ miles, $\angle \text{ABC} = 18° 12′ 14″$, $\text{BAC} = 71° 37′$, $\text{CAD} = 40° 51′$, and latitude 31° 45′.
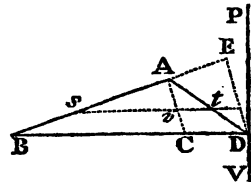
The same answered by Mr. Turner, the Proposer.

Though this question may be solved by a simple quadratic equa-

tion, yet I always prefer conciseness in mathematics, and send you the geometric construction, and trigonometric calculation, which are vastly easier than the algebraic operation.

· *Construction.* Assume any two lines as A*s*, A*t*, in the ratio of 12 to 5, and let them contain the given angle *s*A*t*, produce the lines indefinitely, and let fall A*v* perpendicular to *st*, and at the distance of 230 draw a parallel to A*v*, as V*p*, and from where it cuts A*t* produced as at D, draw D*c* parallel to *ts*, and where it meets A*s* produced will form the triangle required.

Solution.

A*s*, A*t*, being assumed = any numbers in the ratio of 12 to 5, and the angle contained given : Find *st*, and thence *vt* and *v*A. And then it will be as *vt* : *v*A :: CD : CA, from whence all the rest are easily known.

Miles.

The course of one ship is S. 71° 38′ westerly, dist. run 843·7
the other S. 40 52 easterly, ... 351·5
Latitude of the port sailed from 40° north. Difference of latitude 266 miles.

VI. QUESTION 174, *by Mr.* Richard Lovatt.

Having given the lengths of three right lines, 3000, 2000, and 1500, to find the length of a fourth right line, which together with the three former shall include the greatest possible area ?

Solution.

The original solution given in the Diary being erroneous, as well as another afterwards substituted in its stead, it will be proper to give one that is correct, or rather instead of the original problem, let it be required *to inclose the greatest possible space by any proposed number of straight lines all which except one are given in magnitude.*
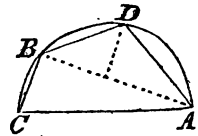
Let ADBC be the maximum figure, of which the sides AD, DB, BC, are given in magnitude and let AC be the side to be determined. From the ends of AC draw lines to B, any angle of the figure ; then as the triangle ABC may vary without changing the magnitude of the remainder of the figure, *viz.* by the part BDA revolving on B as a centre, while the lines BA, BC retain the same magnitude, it is evident that the figure can only be a maximum when the triangle is a maximum, which will be the case when CBA is a right angle. Hence it follows that when the figure is a maximum lines drawn from A and C to each of its angles *must contain* right angles, therefore whatever be the number of its sides *it will always*

be inscribed in a semicircle of which the side to be found is the di-
ameter.

When the figure is a quadrilateral as in the Diary question, let its known sides AD $= a$, DB $= b$, BC $= c$, and let its unknown side AC $= x$. Join AB, CD, then, because by the nature of the circle BC $\times$ AD $+$ BD $\times$ AC $=$ AB $\times$ CD, if we remark that AB $= \sqrt{(x^2 - c^2)}$ and CD $= \sqrt{(x^2 - a^2)}$, we have $ac + bx = \sqrt{(x^2 - c^2)} \times \sqrt{(x^2 - b^2)}$, this equation when cleared of the radicals becomes

$$x^3 - (a^2 + b^2 + c^2)\, x = 2abc.$$

or in numbers $x^3 - 15250000x = 18000000000$.

Hence $x = 4398\cdot03757145$.   L.

<center>THE PRIZE QUESTION, <i>by Mr.</i> Fearnside.</center>

There is a meadow in the form of a right-angled triangle, the base is 95·23 chains, but the other two sides being overflowed with water cannot be measured, however it was observed that a ditch which ended at the acute angle at the base, and crossed the meadow, made with the hypothenuse an angle of 15°; there is also a tree growing exactly in the middle of that part of the perpendicular intercepted between the ditch and the right angle, and it is known that a right line from the tree to the acute angle at the base divides that angle into two equal parts.   Required the other leg or perpendicular of the triangle?

<center><i>Answered.</i></center>

Given BC $= 95\cdot23$ chains $= b$, sine $\angle$ABD $= 15° = s$, $\angle$ABE $=$ EBC, and DE $=$ EC. Put $x = s$. $\angle$EBC; then

$\sqrt{(1 - xx)} : b :: x :$ CE $= \dfrac{bx}{\sqrt{(1 - xx)}}$; hence

$\dfrac{2bx}{\sqrt{(1 - xx)}} =$ DC. Then, per Euc. 47. I. BD $= b$

$\sqrt{\dfrac{3x^2 + 1}{1 - x^2}}$. Again the cosine of double the

$\angle$EBC $=$ sine of the $\angle$A will be found to be $=$

$1 - 2x^2$; then $1 - 2x^2 : s :: $ BD : BA $= \dfrac{bs}{1 - 2x^2}$

$\sqrt{\dfrac{3x^2 + 1}{1 - x^2}}$. Then by 3d prop. 6 Euclid, BC : BA :: EC : AE; which analytically expressed, and the equation ordered becomes $4x^6 - 3s^2x^2 - ss = 0$. Reduced, the sine of the $\angle$EBC will be found to be that of 34° 27′ 47″, and CA $= 247\cdot13$; consequently the area of the meadow will be $= 1176\cdot709$ acres.

<center>P 4</center>
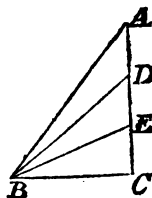
Questions proposed in 1734, *and answered in* 1735.

I.   QUESTION 175, *by Mr.* Rob. Fearnside.

A lady of wit, youth, and beauty beside,
Remote from all cares, but of being a bride,
Surpriz'd her fond lover one morning in May,
And dispatch'd him for parson and licence away.
But how great her confusion, when Strephon brought news
That the parson a licence to grant did refuse!
Her age, which the lady nor lover cou'd tell,
Was the cause that this fatal disaster befel.
    Therefore, ladies, their humble request is, you'll show
The way how to do't from the * data below.
And this by a general method explain,
So that lovers may never be non-plus'd again.

* Cycle of ☉ 18, golden number 8, Roman indiction 10, that year the lady
was born.

Answered by Mr. Richard Dunthorne.

The solution to this question is in Keil's Astronomy, lect. 29, p.
379 and 380, by finding three numbers $285x$, $420y$, and $532z$; so
that the first divided by 28 leaves the cycle of the sun a remainder,
the second by 19 leaves the golden number, the third by 15 leaves
the indiction.   Then if the sum of these numbers be divided by 7980,
the remainder will be the year of the Julian period required.   Or,

The cycle ☉ 18 mult. by 4845 = 87210 ⎫
Golden numb. 8 ....4200 = 33600 ⎬ The sum of the pro-
Indiction   10 ....6916 = 69160 ⎭ ducts is 189970.

Which divided by 7980, there remains 6430, the Julian period; from
which subtract 4713, the Julian period at our Saviour's birth, remains
1717, the year required in answer to the question.

Answered by Mr. John Ommanney.

Dr. Keil in his Astron. Lectures, p. 380, says, if 4845 be multi-
plied by the cycle of the sun, and 4200 by the golden number, and
6916 by the Roman indiction, and the sum of their products divided
by 7980, the remainder (neglecting the quotient) will be the year of
the Julian period; from which subtract 4713, there will remain the
year of the christian æra.   And in this question the answer 1717,
as above.

Additional Solution.

The solar cycle is a period of 28 years, the lunar of 19, and the indiction a period of 15. The year before the christian æra was the 9th of the solar cycle, the 1st of the lunar, and the 3d of the indiction cycle. Wherefore 9, 1, and 3 being severally added to any year x of Christ, and the sums divided respectively by 28, 19, and 15, the remainders will shew the several years of the cycles for that year. But, in the present case proposed, the remainders are 18, 8, and 10;

hence then $\dfrac{x+9-18}{28}$, $\dfrac{x+1-8}{19}$, and $\dfrac{x+3-10}{15}$ must be integers; or $\dfrac{x-9}{28}$, $\dfrac{x-7}{19}$, and $\dfrac{x-7}{15}$ integers. Put the first of these equal to the integer m, that is $\dfrac{x-9}{28} = m$; then $x = 28m + 9$ a value of x answering the first condition. Write this in the 2d,

then $\dfrac{28m+9-7}{19}$ or $\dfrac{28m+2}{19} = m + \dfrac{9m+2}{19}$ = an integer;

therefore $\dfrac{9m+2}{19}$ or $\dfrac{18m+4}{19} = m - \dfrac{m-4}{19}$ = an integer;

therefore $\dfrac{m-4}{19} = n$ an integer; hence $m = 19n + 4$; which substituted in the value of x, we have $x = 28 \times (19n + 4) + 9 = 532n + 121$ a value of x answering the two first conditions. This value of x being written in the 3d original integer, we have $\dfrac{532n + 121 - 7}{15}$

or $\dfrac{532n + 114}{15} = 35n + 7 + \dfrac{7n+9}{15}$ = an integer; hence $\dfrac{7n+9}{15}$

or $\dfrac{14n + 18}{15} = n + 1 - \dfrac{n-3}{15}$ = an integer; therefore $\dfrac{n-3}{15}$ $= p$ an integer; consequently $n = 15p + 3$; this written in the last value of x, it becomes $x = 532 \times (15p + 3) + 121 = 7980p + 1717$ a general expression for the year answering all the three conditions, in which p may be either nothing or any whole number. In the present case it is evident that the value of p must be nothing, and then $x = 1717$, the year of the lady's birth.

It is evident that such a combination cannot happen again till the year $9697 = 7980 + 1717$, when the value of p is 1; and that the successive years of its happening are found by the continual addition of 7980. H.

II.   QUESTION 176, *by Mr. John Gundy.*

Whilst I was surveying for his Grace the Duke of Buccleugh, it was my chance to meet with a piece of land in the form of a rectangle, or long square; and the proportions were such, that if it had been two perches broader, and three longer, it would have been sixty-four perches larger than before. But, on the contrary, if it had been three perches broader, and two longer, it would then be sixty-eight perches larger than it was by my survey. Quere, what was the area of the said piece of land?

Answered by Eumenes Pamphilus.

Let $x =$ length, $z =$ breadth. Then per quest.
$$xz + 3z + 2x + 6 = xz + 64, \text{ and}$$
$$xz + 2z + 3x + 6 = xz + 68. \text{ Their difference is}$$
$x - z = 4$, hence $x - 4 = z$; which being substituted for z, there comes out, by the 1st equation,
$$xx + x - 6 = xx - 4x + 64, \text{ or } 5x = 70. \text{ Hence}$$

$x = \frac{70}{5} = 14$ perches the length, and

$z = x - 4 = 10$ perches the breadth, also
$xz = 140$ perches, or 3 roods 20 perches the content.

Mr. *Turner*'s answer is in substance the same.

III.   QUESTION 177, *by Mr.* John Turner.

Required the greatest cylinder that can be inscribed in a frustum of a cone whose dimensions are as follows: the diameter of the greater end 44 inches, that of the less end 20, and the length 30 inches. Also suppose the frustum to be cut by a plane, parallel to the axis, through the extremity of the less end. What is the content of each of the parts so formed?

Answered by the Proposer.

In order to find the height of the whole cone mn. Put $nm = x$,

c$m = b$, m$n = c$, a$n = d$. Then, by similar triangles, as x : b :: $x + c$: d.

Hence $bx + bc = dx$, and $x = \dfrac{bc}{d-b}$

$= 25$; consequently B$n = 55$ inches. Again, put $p =$ Bn, $b =$ AP, $d = \cdot 7854$, $x = n w$ the height of the greatest cylinder;

as p : b :: $p - x$: $\dfrac{bp - bx}{p} = pq$. But the solidity of the cylinder $=$

$\dfrac{bbxx - 2bbpx + bbpp}{pp} \times dx$; in fluxions,

it is $3dbbxx\dot{x} - 4dbbpx\dot{x} + dbbpp\dot{x} = 0$. Divide all by $dbb\dot{x}$, and $3xx - 4px + pp$
$= 0$; Ergo $x = \frac{1}{3} p = 18 \cdot 3333$ inches, the cylinder's height; and the diameter of its base 29·3333. Again, from A$g = 12$, and P$g = 32$, being given, $gv = ge$ is found $= (\sqrt{12 \times 32})$ 19·6 inches, and $ve = 39 \cdot 2$: the area of the segment Aev (by page 404 of Ward, or much easier by a table of segments) is found to be 335·88 square inches. Which multiplied by $\frac{1}{3}$B$n = 18 \cdot 333$, gives the solidity of the pyramid ABevA $= 6157 \cdot 8$ cubic inches.

Now the base of the pyramid CvevBC is an hyperbola, whose area is thus found: $ng = 10$ is the hyperbola's conjugate semi-diameter $= b$, $ve = 39 \cdot 2 = g$ its bounding ordinate, $cg = 30$ the abscissa, cq the transverse semi-diameter $= 25$, Q$g = p = 55$, AP $= d = 44$. The

hyperbolic logarithm of $\dfrac{d + g}{2b}$ or 4·16 is 1·4255142 $= s$. I say the

area of the hyperbola is $= \dfrac{pg}{2} - \dfrac{2pbbs}{d} = 721 \cdot 62$ square inches,

which is a contraction of Dr. Wallis's quadrature of the hyperbola, p. 328, of his Algebra. This multiplied by $\frac{1}{3}ng$, the perpendicular altitude of the pyramid, or 3·333, gives 2405·40 its solidity. Subtract this pyramid CvevBC from the pyramid ABevA, the remainder is the content of the ungula or cuneus AvecA $= 3752 \cdot 4$ cubic inches, which was required. And lastly, if you subtract this from the content of the conic frustum ACDP, which is easily found $= 25258 \cdot 46$, the remainder is the solidity of the other part or ungula PevcP $= 21506 \cdot 06$ inches.

Answered by Mr. Geo. Brown.

Who projects the scheme, and all its parts exactly the same as above, and carries the process through the whole, from whence I shall only collect such parts as are essential to the answer.

The area of the segment A$ev = 335 \cdot 915$ square inches,

The solidity of the inclined pyr. AB$g = 6158 \cdot 45$ cubic inches,
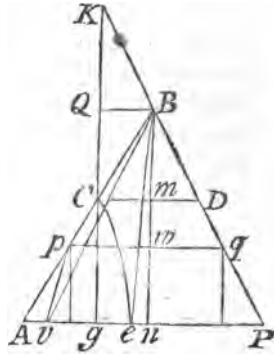
The area of the hyperbola $= 721 \cdot 45$,

Solidity of the inclined pyramid c$_{Bg} = 2404 \cdot 95$,

Which taken from A$_{Cg}$,

Leaves the solidity of the hoof A$_{Cve} = 3753 \cdot 49 = 2 \cdot 1721$ feet.

Which taken from the given frustum of the cone $14 \cdot 6171$,

Leaves the content of the part P$_{DCrep} = 12 \cdot 445$.

The height of the greatest cylinder $mn = 18 \cdot 333$ $\Big\}$

And its base's diameter............. $29 \cdot 333$ $\Big\}$ inches.

IV. QUESTION 178, *by Mr.* Chr. Mason.

What is the solid content of a piece of timber bounded by four equal planes, each of which is an isosceles triangle, the longest side 5 and the shortest 3 feet?

Answered by Juvenis Mathematicus.

This body seems to agree with Euclid's definition of a pyramid Per 47 Euc. I. $cc - \frac{1}{4}bb = \frac{1}{4}(4cc - bb) =$ to the perpendicular of one of the isosceles triangles; and if the body be conceived to be divided into two equal parts by a triangular plane, two sides of which are the perpendiculars above, and the 3d $= b = 3$. Then by the 14th prop, of Keil's Plain Trigonometry, $\frac{1}{2}\sqrt{(4cc - bb)} : \frac{1}{2}\sqrt{(4cc - bb)} + b :: \frac{1}{2}\sqrt{(4cc - bb)} - b$: the difference of the segments of the base. The rectangle of the means divided by the first extreme $= \frac{1}{4}\sqrt{(4cc - bb)} - bb \div \frac{1}{2}\sqrt{(4cc - bb)} =$ the difference of the segments. Hence then $\frac{1}{2}\sqrt{(4cc - bb)} - bb \div \sqrt{(4cc - bb)} =$ the greater segment, and $bb \div \sqrt{(4cc - bb)} =$ the less segment; also $bb - b^4 \div (4cc - bb) =$ the square of the perpendicular height of this triangle. But $\frac{1}{2}b\sqrt{\dfrac{4cc - bb}{4}} =$ area of one of the isosceles triangles or base of the solid body. Wherefore $\frac{1}{3}\sqrt{\left(bb - \dfrac{b^4}{4cc - bb}\right)}$

$\times \frac{1}{2}b\sqrt{\dfrac{4cc - bb}{4}} = 6 \cdot 791536746428$ cubic feet, the solid content required.

The same answered by Mr. Ed. Golding.

A general Rule to find the Solidity. From the square of one of the legs, abate the square of half the base; and multiply the square root of the remainder by the square of the base: The last product divide by 6, and the quotient shall be the content of the proposed solid.

For the perpendicular Height. The square of the side or leg $5 =$

25; subtract twice the square of half the base 1·5 = 4·5; there remains 20·51 whose square root 4·5277 is the perpendicular height.

For the Solidity. Multiply the square of the base 3 = 9, by the perpendicular 4·5777, and the product 40·7493 is the solidity of the circumscribing prism. As 6 : 1 :: 40·7493 : 6·79155 the solid feet in the alabrum, by which name I call this solid till a fitter name be imposed. And to prove it but $\frac{1}{6}$ of the prism. Suppose the first figure, being a square prism, be 3 feet square at the ends, and 6 feet high, the content will be 54. Cut off from the prism (1.) the two prismatic wedges a and b; which both together contain 9 upon 3 = 27 feet; leaving the wedge c, which suppose to be the second figure; from it cut off the two pyramids e and f, which both together contain 9 upon 2 = 18. Leaving the alabrum only behind: The two wedges = 27, and two pyramids = 18, make 45, which taken from 54, remains the alabrum 9 = $\frac{1}{6}$ of the prism. *Q. E. D.*

The same answered by Mr. J. Bulman.

From the square of the given side of the isosceles triangle = 25, subtract the square of half the given base = 2·25, the remainder 22·75 is the square of the perpendicular to the triangle: from which subtract again the square of half the base, the square root of the remainder 4·527692 = the length of the circumscribing square prism: whose solidity is 40·749228 feet, $\frac{1}{6}$ of which is 6·791538 feet, the solidity of the body required.

This solid may easily be cut from a prism having square bases. Or if a piece of stiff paper or pasteboard be cut according to the following figure, and folded in the lines ab, bc, and ca, so as cf and ce coincide or make but one line, and g touches the joined points e and f, it will perfectly represent the body proposed in the question.
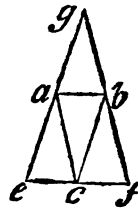
v. QUESTION 179, *by Mr. J. Bulman.*

At a certain place in northern latitude, the sun was observed to rise exactly at 3^h 58^m and at 6 o'clock his altitude was taken the same morning, and found to be 15 deg. 20 min. his declination being then north. Required the latitude of the place where, and day of the year when, those observations were made?

Answered by Mr. N. Oats.

Put $a =$ sine of the altitude at 6; $d =$ sine of the ascen. diff. $s =$ sine of the latitude; $\sqrt{(1 - ss)} =$ its cosine. Then, per spherics, $s : 1 :: a : \dfrac{a}{s} =$ sine decl. and its cosine $= \sqrt{\left(1 - \dfrac{aa}{ss}\right)}$

$= \dfrac{\sqrt{(ss - aa)}}{s}$. Again $\dfrac{\sqrt{(ss - aa)}}{s} : \dfrac{a}{s} :: 1 : \dfrac{as}{\sqrt{(ss - aa)}} =$

tang. declin. $\sqrt{(1 - ss)} : s :: 1 : \dfrac{s}{\sqrt{(1 - ss)}} =$ tang. lat. 1 :

$\dfrac{s}{\sqrt{(1 - ss)}} :: \dfrac{as}{\sqrt{(ss - aa)}} : d;$ ergo $\dfrac{ass}{\sqrt{(ss - aa)}} = d\sqrt{(1 - ss)}.$

Reduced, it is $s^4 - aass + aa - ss + \dfrac{aass}{dd} = 0$; hence $s = \cdot 835728$

$=$ the sine lat. $= 56° \ 41' \ 32''$. And the declination $= 18° \ 26' \ 45''$, answering to May 2, or Aug. 4.

Additional Solution.

Let x and $y =$ sine and cosine of the latitude,
  v and $z =$ sine and cosine of the declination,
  $s =$ sine of the altitude at 6; and
  $a =$ sine of the ascensional difference.

Then per spherics rad. $: \dfrac{x}{y} :: \dfrac{v}{z} : a$, and rad. $: v :: x : s;$

∴ $vx = azy$ and $vx = s$, and ∴ $azy = s$ or $zy = s \div a$.

Therefore by addition and subtraction $zy + vx$ the cosine of the difference of the latitude and declination, and $zy - vx$ the cosine of the sum of the latitude and declination are known, and consequently the sum and difference of the latitude and declination are known. Half the sum is the latitude and half the difference is the declination. **L.**

VI. QUESTION 180, by Mr. John Gundy.

A gentleman, who was a great lover of the mathematics, had a large estate, which lay in four several entire manors, of which he had drawn four several maps. Now he, by his often looking over the contents of these maps, found the quantities of acres belonging to each of these, would make four numbers in continued proportion; whereof the sum of the two middlemost numbers is 1152, and that of the two extremes 1728. He lying on his death-bed, called his four sons to him, saying, My dear children, my glass being almost run, and the estate I have to leave amongst you was for the most part gained by my own industry, and so entirely at my own disposal,

I shall leave it amongst you with this proviso, that he that answers the abovesaid question first, shall have the largest manor for his share; and so the rest of you the others, according to your birth-right. It is demanded the quantity of acres that each manor does contain?

<p align="center">*Answered by Mr. Beacham.*</p>

This question is composed out of Sir Isaac Newton's Arith. or Ronayne's Algebra. The first share is 1536, the second 768, the third 384, and the fourth 192 acres.

<p align="center">*First Solution.*</p>

Let v, x, y, and z denote the numbers sought, and put $a = 1728$ and $b = 1152$. Then by the question,

$$v + z = a, \qquad x + y = b,$$

and by the nature of continued proportion

$$v : x :: x : y, \text{ or } vy = xx,$$
$$\text{and } x : y :: y : z, \text{ or } xz = zz.$$
$$\text{But } z = a - v \text{ and } y = b - x;$$

therefore $v(b - x) = xx$ and $x(a - v) = (b - x)^2$.

$$\text{Therefore } v = \frac{xx}{b - x} \qquad v = \frac{ax - (b - x)^2}{x};$$

and therefore $x^3 = ax(b - x) - (b - x)^3$,

$$\text{Reduced } \quad x^2 - bx = -\frac{b^3}{3b + a}.$$

Hence $x = \frac{b}{2} \pm \frac{b}{2} \sqrt{\frac{a - b}{3b + a}} = 576(1 \pm \tfrac{1}{3}) = 768$ or 384.

And the four shares are 1536, 768, 384, and 192 acres.   L.

<p align="center">*Second Solution.*</p>

Let the numbers sought be denoted by x, rx, r^2x, and r^3x, which numbers are evidently in continued proportion, and put $a = 1728$ and $b = 1152$: then by the question $x(1 + r^3) = a$ and $x(r + r^2) = b$. Multiply the second equation by r and subtract the product from the first, and there results $x(r^2 - 1) = br - a$; but from the second equation $x = \frac{b}{r + r^2}$, therefore by substitution $\frac{b}{r + r^2}(r^2 - 1) = br - a$, or $\frac{b}{r}(r - 1) = br - a$: whence $br^2 - (a + b)r = -b$, and by completing the square, &c.

$$r = \frac{a+b}{2b} - \sqrt{\left(\frac{(a+b)^2}{4bb} - b\right)} = 2.$$

Whence $x = 192$, $rx = 384$, $r^2x = 768$, and $r^3x = 1536$, as before. L.

THE PRIZE QUESTION, *by Mr.* Mason.

A gentleman wants a garden of ten acres, of an oblong form, to embellish the front of his house. In the middle there is to be an elliptical fountain, 3 feet deep and 20 rods in area: the principal axes of the ellipse to have the same proportion as the length and breadth of the garden, and the transverse axis to be parallel to the longest side. The garden is also to be surrounded on three sides by a canal, the breadth and depth of which are to have the ratio of 7 to 1; the earth from the canal together with that from the fountain, to be spread over the garden and to raise it a foot. The nearest distance from the edge of the fountain to the edge of the canal to be 238 feet. Required the dimensions of the garden, fountain, and canal?

Answered by Mr. J. Doubt.

Given 10 acres $= 435600$ square feet $= k$; 20 rods $= 5445$ square feet $= g$; the distance of the fount to the canal $= 476 = b$: The area of a circle whose diameter is $1 = \cdot7854 = m$: Put the transverse diameter of the fount $= x$, and the conjugate $= y$: then

$\frac{g}{xm} = y$; $y : x :: b : \frac{bxxm}{g} = a$: and also $(a + x) \times (b + y)$

$= \frac{bbxxmm + 2bxgm + gg}{gm} = k$: and so $bxm + g = \sqrt{kgm}$.

Hence will be found $x = 100\cdot8844$, $y = 68\cdot7199$. The length of the garden $799\cdot677$ feet; breadth $544\cdot719$ feet; and then $435600 - 5445 - 16335 = 413820$ the solid content of the canal, which let be $= s$, and $799\cdot6771 \times 2 + 544\cdot719 = 2144\cdot074$ (the extent of the canal) $= l$. Put breadth $= a$, the depth $= e$; then per question, $7 : 1 :: a : \frac{a}{7} = e$; and $\frac{laa}{7} = s$. Therefore $a = \sqrt{\frac{7s}{l}}$ $= 36\cdot7566$ feet; and $e = 5\cdot2509$ feet.

Questions proposed in 1735, *and answered in* 1736.

1.   QUESTION 181, *by Mr.* John Turner.

In the triangle ABC there is given the side AB $=$ 65 feet, and the side BC $=$ 74; and in this triangle there is an equilateral triangle inscribed as SWH, whose side SH is parallel to the base AC : And lastly, in the equilateral triangle there is a circle inscribed, whose area is known to be $=$ 307·9 square feet.   It is required to find the side of the equilateral, and the base of the external triangle?

Answered by Mr. Christ. Mason.

First, per Euclid, circles are in proportion as the squares of their diameters, therefore the diameter of the circle will be found 19·79976.   And, per plain trigonometry, as s. 30° : diameter :: s. 60° : 34·28 the side of the circumscribing trigon $=$ SH $=$ HW $=$ WS.   The square root of $\frac{3}{4}$ of the square of the side is the perpendicular VW $=$ 29·69.
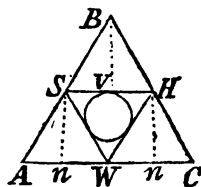
Let $b =$ AB $=$ 65; $c =$ BC $=$ 74; $d =$ SH $= 34\cdot28$; $p =$ VW $= 29\cdot69$; $a =$ BV sought.

Then 4 Eucl. VI. as $a : d :: a + p : \dfrac{da + dp}{a} =$ AC: And per 47 Eucl. I. $\sqrt{(bb - (aa + 2pa + pp))} =$ AW; also $\sqrt{(cc - (aa + 2pa + pp))} =$ WC; and AW $+$ WC $= \dfrac{da + dp}{a}$: which, properly reduced, gives $a = 31\cdot42$; and $a + p =$ BW $= 60\cdot735$; AW $= 23\cdot15$; WC $= 42\cdot28$; therefore AC $= 65\cdot43$.

The same answered by Eumenes Pamphilus.

As 3·1415926 &c. : 1 :: 307·9 : 98·60763 $=$ sq. radius.   Whose root 9·89988 $=$ radius; and 19·79976 $=$ diameter. Then, by Ward's Proportion, p. 336, as ·28867513 : 2 :: 9·899 : 34·29419 the side of the equilateral triangle required.

Now let BA $= a = 65$; BC $= b = 74$; SH $= c = 34\cdot29419$; $x =$ AC; and let sn and Hn be perpendiculars let fall from the points s and H upon the base AC. Then will $sn =$ H$n =$ three times the radius of the inscribed circle $= 29\cdot69964 = d$; then, per 2 Eucl. VI. $x : c$ :: $a : \dfrac{ac}{x} =$ BS; and $x : c :: b : \dfrac{bc}{x} =$ BH; $a - \dfrac{ac}{x} =$ SA; and

$b - \dfrac{bc}{x} = $ HC; then per 47 Euc. I. $\sqrt{}(aa - \dfrac{2aac}{x} + \dfrac{aacc}{xx} - dd) =$

An; and $\sqrt{}(bb - \dfrac{2bbc}{x} + \dfrac{bbcc}{xx} - dd) = nc$; consequently $x = c +$

$\sqrt{}(aa - \dfrac{2aac}{x} + \dfrac{aacc}{xx} - dd) + \sqrt{}(bb - \dfrac{2bbc}{x} + \dfrac{bbcc}{xx} - dd)$; and

by reduction $x = 67{\cdot}743589 = $ AC the base required.

II. QUESTION 182, *by Mr*. Chr. Mason.

A surveyor measured a field in the form of a trapezium, the longest side was 25 chains. The two angles subtended by the given side at the extremities of the opposite side were 118° and 108°; and the perpendiculars from the said angles to the given side were 9 and 7 chains respectively. Required the three unknown sides of the trapezium?

Answered by the Proposer Mr. Mason.

The obtuse angles given are $adb = 108°$ and $aob = 118°$; the perpendiculars $pd = 9$ chains, and $eo = 7$ chains; and the base $ab = 25$ chains. Required bd, do, and oa; the other three sides of the trapezium? Which will be found trigonometrically, *viz.* $ad = 16{\cdot}61$; $db = 14{\cdot}23$; $bo = 17{\cdot}71$; $ao = 11{\cdot}15$; $do = 5{\cdot}76$.
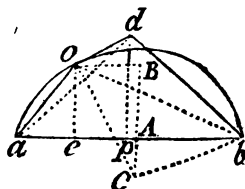
Additional Solution.

In this question are concerned two triangles aob, adb, on the same given base, and whose perpendiculars and vertical angles are also given; to find their sides and the distance of their vertexes.

Construction. On the given base describe two segments of circles capable of containing the given vertical angles; then at distances equal to the two perpendiculars draw two parallels to the base, and they will cut the respective circles in the required vertexes o and d of the two triangles. As is too evident to need a demonstration.

Calculation. Let c be the centre of the circle passing through one of the vertexes; and cAB perpendicular and oB parallel to ab, and the other lines as per figure. Then A$cb = 62° = $ the supplement of the given $\angle aob$, and Aco is $= $ the difference of the $\angle$s (oab, oba) at the base. But s. $\angle Acb$: cos. $\angle$Acb :: A$b(\frac{1}{2}ab)$: Ac; hence cb $=$ A$c + oe$; then (since Ab and cb are the sines of the $\angle$s Acb and

cob to the same radius *cb* or *co*) A*b* : C*B* :: s. ∠A*cb* : s. ∠C*o*B = the complement of the difference of the ∠s *oab*, *oba*, at the base. Hence their sum and difference being known, the angles themselves become known; and thence the sides *ao*, *bo*.——In like manner are found the sides *ad*, *bd*. And thence *od*. H.

III. QUESTION 183, *by Mr.* Tho. Fearn.

A General disposing his army into a square battle, finds he has 284 soldiers over-and-above; but increasing each side with one man, he wanted 25 to fill up the square. Quere, the number of soldiers?

Answered by Russellus.

Put a = side of the first square; b = the overplus (284); and c = 25 the number wanting to fill up the second square. Then $aa + b = aa + 2a + 1 - c$ = the number of soldiers per question ∴ $b = 2a + 1 - c$; $b + c - 1 = 2a$; and $\dfrac{b + c - 1}{2} = a = 154$; $aa + b = 24000$ = the number of soldiers.

The answer by *Eumenes Pamphilus*, being nearly the same as the above, is omitted.

IV. QUESTION 184, *by* Eumenes Pamphilus.

There are three numbers in continued proportion; the product of the three multiplied into one another is 512, and the sum of the extremes 34. Three-fourths of the greater extreme being called years, and the other fourth weeks, will exactly shew my age; the mean, the month, and the less extreme, the day of the month I was born: Whence you are desired to shew the year, month, and day on which I was brought to light? Sep. 27, 1733.

Answered by Mr. Ed. Golding.

The three numbers in continual proportion are 2, 8, 32; for 2 × 8 × 32 = 512, and 2 + 32 = 34; which exactly agrees with the proposal: three-fourths of the greater extreme is 24 for years; and one-fourth is 8 weeks; so he was 24 years 8 weeks old; and he was born August 2, 1709.

The same answered by Mr. John Turner.

Put a, e, and u for the three numbers sought; $b = 512$; $c = 34$.

Then, by the question, $aeu = b$; $a + u = c$; and $au = ee$: by the first $au = \dfrac{b}{e}$ (in the 3d step) $= ee$; hence $eee = b$; and $e = \sqrt[3]{b} =$ 8 the mean. And from hence u is found $= 32$, and $a = 2$; so he was 24 years and 8 weeks old.

Mr. *Mason's* answer is omitted, being nearly the same as Mr. *Turner's*.

<p style="text-align:center;">V.   QUESTION 185, <i>by Mr.</i> Rob. Fearnside.</p>

To determine the greatest area included by the curve of a parabola, whose equation is $ax^2 = y^3$, and a rectangular double ordinate, the length of the curve being 100 feet?

<p style="text-align:center;"><i>Answered by Mr.</i> Feàrnside <i>the Proposer.</i></p>

Putting the equation $ax^2 = y^3$ of the curve into fluxions, we have $2ax\dot{x} = 3y^2\dot{y}$; hence $\dot{x} = \dfrac{3y^2\dot{y}}{2ax} = \dfrac{3\dot{y}}{2}\sqrt{\dfrac{y}{a}}$. Then the fluxion of the area $= y\dot{x}$ is $\dfrac{3y\dot{y}}{2}\sqrt{\dfrac{y}{a}}$, whose fluent $\dfrac{3y y}{5}\sqrt{\dfrac{y}{a}}$ is the area a maximum. Again, the fluxion of the curve $\sqrt{(\dot{x}^2 + \dot{y}^2)}$ will become $\dot{y} \times \sqrt{\dfrac{9y + 4a}{4a}}$, whose correct fluent $\left\{ \left(\dfrac{9y + 4a}{a}\right)^{\frac{3}{2}} - \frac{8}{?} \right\} \times \dfrac{a}{27}$ is $=$ $50 = c$. Then the value of either a or y being found from this equation, and written for it in the above area or maximum, the maximum will then contain only one of the letters a or y, and whose fluxion being made equal to 0, that letter will be determined, and thence all the rest.

<p style="text-align:center;">VI.   QUESTION 186, <i>by Mr.</i> Ed. Hauxley.</p>

The use of a meridian line in astronomy, geography, dialling, &c. is very great, and on its exactness all depends : whence infinite pains have been taken by divers astronomers to have it to the last precision. M. Cassini has distinguished himself by a meridian line drawn on a copper-plate on the pavement of the church of St. Petronia, at Bologna in Italy, the largest and most accurate in the world. In the arched roof of the church, at a great height above the pavement, is a little hole, through which the sun's image when in the meridian, falling upon the line marks his progress all the year. When finished, Mr. Cassini, in a public writing, informed the mathematicians of Europe of a new oracle of Apollo, or the sun, established in a tem-

ple, which might be consulted with entire confidence as to all diffi-
culties in astronomy.

Bologna, by some tables, lies in the latitude of 44° 8' north; and
Dr. Burnet, late bishop of Salisbury, in his letters many years ago,
has given us an account of this meridian line, from whence the height
of the hole in a perpendicular above the pavement, and the length
of that copper meridian from that perpendicular, under the said hole,
to its utmost extent, where the sun's image marks the tropic of Ca-
pricorn, in one sum makes 285 feet 10 inches. From hence the
height of the hole, or nodus above the floor, and the length of the
meridian line, and the distances of the tropics from the equinoxes
may be severally known, and are here required?

Answered by Mr. E. Golding.

There is given AB + BC = 285 feet 10 inches; required AB, BC,
BE, and BD. I here use 23° 30' for the
greatest declination: then 90° — 44° 8'
= 45° 52' the elevation of the equinoc-
tial; and the tropics and equinoctial will
have the same angles in respect to each
other. Of all which BD, BE, BC are tan-
gents to the radius AB. Produce the line
CB till Bd = BA; then, per question, cd
= 285 feet 10 inches = 3430 inches. Then the angle dAB = angle
d = 45°, and (CAB =) 68° 31' + 45° = 113° 31' = CAd.
As s. CAd : cd :: s. d : CA = 2649·24 inches.
Rad. : AC :: s. CAB : CB = 2453·95 = 204 feet 5 inch. 95 par.
Rad. : AC :: s. ACB : AB = 976·05 = 81 feet 4 inch. 05 par.
                                                                    ——   ——   ——
                                    Sum 3430 = 285   10   0
                                                                    ——   ——   ——
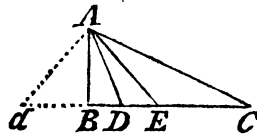
And proceeding in the same manner we shall find

	Inches	Feet	Inch.	Parts.
BD = ..,....	380·7423 =	31	8	7423
♋ ♎ DE	588·8556 =	49	0	8556
♈ ♑ EB	1484·3521 =	123	8	3521
The sum = BC = ...	2453·95 =	204	5	95
To which add AB = ...	976·05 =	81	4	05
Total.....................	3430· =	285	10	0

N. B. No notice is taken of the distance of Bologna's longitude in
time from the ☉'s entering the first scruple of the equator or tropics,
nor of the ☉'s refraction, in the answer to this question, as being too
curious.

Q 3

Additional Solution.

The above calculation of this question is not very accurate with regard to the angles; however any one may easily make it as accurate as he pleases by the same method, or rather by this following.

In the above figure, we have given all the angles formed at the point A, and the sum of the legs AB, BC; to find the rest. For A represents the hole, BC the meridian, AB the perpendicular on it, AE a ray from the sun in the equinoctial, and AD, AC rays from him when in the two tropics : therefore the $\angle$BAE $=$ the lat.$=44°$ 8$'$; to and from which adding and subtracting the declination $(23° 29')=\angle$EAC $=\angle$EAD, the sum $(67° 37')$ will be the $\angle$BAC, and the difference $(20° 39')$ the $\angle$BAD.

But, by plain trigonometry, 1 (rad.) : t (tang. $\angle$BAC) :: AB : BC;

then, by composition, &c. $1 + t$; AB $+$ BC :: $\begin{cases} 1 & : \text{AB}, \\ t & : \text{BC}. \end{cases}$

Which being known, then 1 : AB :: $\begin{cases} \text{tang. } \angle \text{BAE} & : \text{BE}, \\ \text{tang. } \angle \text{BAD} & : \text{BD.} \quad \text{H.} \end{cases}$

THE PRIZE QUESTION, *communicated by Mr.* Samuel Ashby, *in a Letter to* Tho. Grubbian.

If in any plain triangle, as ABC, you draw lines from each angle through any point E (o) within the triangle, till they cut the opposite sides, as CA, AB, and BC. The rectangles of the alternate segments at those sides will be equal.

$$\text{viz. } Ab \times ca \times Bc = bc \times aB \times aA,$$

The demonstration of this curious proposition is required?

Answered by Mr. C. Mason.

Per 31 Eucl. I. draw lines parallel to the given sides through o the given point, as *dg, eh, fi*. Then, per 2 Eucl. VI. will be the following proportions, viz. *do* : *og* :: A*b* : *bc*; and *io* : *of* :: A*c* : *c*B; also *eo* : *oh* :: B*a* : *ac*; from which will proceed the following compound proportions, viz. *do* $\times$ *io* $\times$ EO : *og* $\times$ *of* $\times$ *oh* :: A*b* $\times$ A*c* $\times$ B*a* : *bc* $\times$ *c*B $\times$ *ac*; and, per 4 Eucl. VI. *ho* : *eo* :: *io* : *de*; and *og* : *od* :: *of* : *de*;

Ergo . $\dfrac{eo \times io}{ho} = \dfrac{do \times of}{og}$ which, reduced,

gives *do* $\times$ *fo* $\times$ *ho* $=$ *eo* $\times$ *io* $\times$ *go*; permt. alterna. & comp. ergo A*c* $\times$ B*a* $\times$ *bc* $=$ *c*B $\times$ *ac* $\times$ *b*A. Q. E. D.

Mr. J. Turner's answer is omitted, as being nearly the same.

The letter o is omitted at the point of intersection.
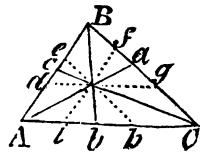
Another Demonstration by Mr. Ed. Enotts.

Draw xcz parallel to AB, and produce Ab and BC till they meet
with the line xz. The △ AEa is similar to CEz;
and xCE to aEB; and bzc to AbB; also $x$cc to ACB:
therefore xc : CE :: Ba : aE; and CE : Cz :: aE
: aA; also xc : Cz :: Ba : Aa; but xc : Cz is
compounded of xc : AB and of AB : Cz; but
xc : AB :: CC : CA, and AB : Cz :: Bb : bC;
therefore xc : Cz (:: Ba : Aa) :: CC : CA and
Bb : bc; that is Ba : Aa :: CC × Bb : CA × bc;
therefore Ba × AC × cb = Aa × CC × Bb.   Q. E. D.
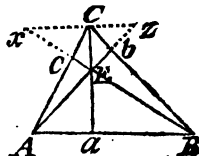
Questions proposed in 1736, and answered in 1737.

I.   QUESTION 187, by Mr. Tho. Simpson.

By reading your di'ry a croud of strange notions
Crept into my head, of your rules, laws, and motions;
Your extravagant fancies my senses confound;
Can the unwieldy-earth at the sun caper round?
But you say, she's an atom, each star a huge sun,
And attendant worlds with their moons round 'em run.
Such a tott'ring strange whirligig you've set's upon,
We wonder ere now we're not shak'd off and gone :
If what eyes ne'er saw you so soon can disclose,
Then pray solve this question; The earth, we'll suppose,
Round her axis in thirty-eight minutes to roll ;
Shou'd we, who're degrees thirty-eight from the pole,
Be hurl'd thro' the air; where should we descend?
How long wou'd it be ere our circuit did end?
How far from the centre in six hours time
Wou'd they be, who live in the midst o'th' hot* clime?
  Kind artist, be pleas'd these things to let's know?
  We'd rather believe you, than e'er find them so.

* The equator.

N. B. We suppose the earth sole actor, and to continue inviolate,
and that we shall acquire the same velocity as the place of our re-
sidence.   52 deg. lat.

Answered by Mr. J. Turner.

The circumference of the earth being 25080 miles, at the equator, if it revolves round its axis in 38 minutes, each point of the equator will go through 11 miles each second, with this velocity the body is projected from P. Let $sp = 1$: then the velocity of a body moving in a circle pq, at the distance of one semi-diameter from the earth's centre, is such, as would make it go 4·92 miles each second uniformly; and according to Mr. Abr. de Moivre, in Philos. Transact, Abridged, p. 5. vol. 4.
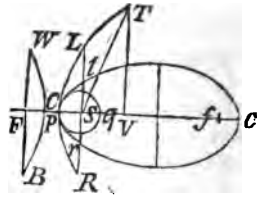
Putting $R = 11$, $Q = 4.92$; $RR - 2QQ$: RR :: SP : PF. *i. e.* 72·6 : 121 :: 1 : 1·666, and the point F will be the other focus; and because $FC = PS = 1$, therefore $CP = ·666$ the transverse axis of the section, which is an hyperbola.——QQ : RR :: $2SP$: LR; that is, 24·2 : 121 :: 2 : 10; hence the latus-rectum of the hyperbola is 10 semi-diameters of the earth.

Again, each point in the parallel of 52 degrees goes through 6·773 miles each second, put this $= R$; $2QQ - RR$: RR :: SP : pf; that is, 2·53 : 45·87 :: 1 : 18·13, and the point f will be the other focus: and because $fc = PS = 1$, therefore $CP = 19.13$, the transverse axis of the section, which in this case is an ellipsis, and QQ : RR :: $2SP$: $lr = 3.791$ its latus rectum.

Merones says that the body in 6 hours will be 8 of the earth's diameters from the centre. Mr. Lowe says the equator will go 10·9 miles, the parallel of 52°, 6·7; and in 6 hours the moving body would be 1486709 miles from the earth's centre. Mr. Abr. Lord 246321, Mr. Geo. Brown 204660, and Mr. Hauxley 400589 miles. I have not the ingenious author's solution by me, to the intricate and difficult question, so shall say no more of it till I can procure it,

Additional Solution,

The above computations of Mr. Turner's are rightly made as far as they are carried, but he has left some part of the question undetermined, which we shall here supply.

The body projected from the latitude of 52° will describe an ellipse whose focus is the centre of the earth, its transverse axe $= 19·13$ (semi-diameters of the earth), and its parameter 3·791, as determined above. So that the ellipse, no where touching the earth but in the place from whence the body is projected, the body will revolve about the earth without falling upon it again, or touching it except in the place from whence it was projected, supposing the earth's revolution to cease at the instant when the body is projected; but if the revolution be contiuned, the body will touch some other part of the same parallel of latitude, or will return to the vertex of its orbit again when some other point of the said parallel passes under it in revolv-

ing; and which point will be thus found: Putting $s = 16\frac{1}{12}$ feet $v = 6.773 \times 5280$ feet velocity in the vertex, and $f = sv = 3992 \times 5280$, feet; then the periodic time, or time of one revolution, is known to be $3.1416 \times \dfrac{4sf}{(\frac{1}{4}fs - vv)^{\frac{3}{2}}} = 132277$ seconds; this being divided by (38′ or) 2280″, the quotient 58 shews the number of compleat revolutions the earth makes in the same time, and the remainder 37 shews what part of another revolution is made; wherefore as 2280 : 37 :: 360° : 5° 50′ = the difference of longitude required from the point of projection.

Again, with regard to the body projected from the equator, the path will be an hyperbola whose focus is also the earth's centre, its transverse axe $\dfrac{2}{3}$, and latus-rectum 10, as determined above; and consequently its conjugate axe $= 2\sqrt{\dfrac{5}{3}}$. Then to find at what distance from the focus it will be in 6 hours $= 21600$ seconds: Since the body describes equal areas in equal times, and the rate of description being $\dfrac{11 \times 1}{2 \times 3992} = \dfrac{11}{7984}$ square semi-diameters of the earth per second, therefore $\dfrac{21600 \times 11}{7984}$ will be the area described in 6 hours, or the area included by the curve, and the focal distances from the vertex and the other end of the curve, viz. the space sPT, and in which Ps $= 1$.

Now if the ordinate TV be drawn, and PV be put $= x$, the transverse PC $= \dfrac{2}{3} = t$, the conjugate $2\sqrt{\dfrac{5}{3}} = c$, and the area PTS $= \dfrac{21600 \times 11}{7984} = A$; then SV $= x - 1$, TV $= \dfrac{c\sqrt{(tx + xx)}}{t}$, and the $\triangle$ STV $= \dfrac{x - 1}{2} \times \dfrac{c\sqrt{(tx + xx)}}{t}$; consequently the hyperbolic segment will be expressed by $A + \dfrac{x - 1}{2} \times \dfrac{c\sqrt{(tx + xx)}}{t}$. But by Rule IV. p. 376 Mensuration, the area of the same segment is expressed by $\dfrac{21\sqrt{(tx + \frac{5}{7}xx)} + 4\sqrt{tx}}{75} \times \dfrac{4cx}{t}$; these two expressions being made equal to each other, and reduced, we have $168 \times x\sqrt{(tx + \frac{5}{7}xx)} + 32\,x\sqrt{tx} = \dfrac{150At}{c} + (x - 1) \times 75\sqrt{(tx + xx)}$; or, by restoring the values of t and c, it will be $168\,x\sqrt{(2x + 2\frac{1}{2}xx)} + 32\,x\sqrt{2x} = 30A\sqrt{5} + (x - 1) \times 75\sqrt{(2 + 3xx)}$; and the root x is easily found $= 3.083 = $ PV. Then ST $= \sqrt{(TV^2 + VS^2)} =$

$$\sqrt{\left(cc \times \frac{tx + xx}{tt} + (x-1)^2\right)} = 4x + 1 = 13\tfrac{1}{4} \text{ semi-diameters}$$

$= 53227$ miles, the distance required. H.

II. QUESTION 188, *by* Eumenes Pamphilus.

Find the third side of a triangular field, whose area is 138 acres, and two of its sides 296 and 204 poles?

Answered by the Proposer.

Let $x =$ the side sought, then will $\tfrac{1}{2}x + 250 =$ half the sum of the sides; each side being subtracted from it, and the three remainders multiplied into it, will give the square of the content, and will produce $16154\,xx - \tfrac{1}{16}x^4 - 132250000 = 487526400$ poles; which, reduced, gives $x = 460$ poles, the side required.

Mr. Turner's solution is omitted, as being on the same principles.

Additional Solution.

Let the two given sides be a, b, and the given area A. Then $2\text{A} \div ab =$ the sine of the included angle, whose cosine call c; then, by trigonometry $\sqrt{(aa + bb - 2abc)}$ or $\sqrt{(aa + bb - 2\sqrt{(aabb - 4\text{A}^2)})}$ $=$ the side required. H.

III. QUESTION 189, *by Mr.* Rob. Beighton.

Solomon, we are assured in Holy Writ, was a man of the most extensive judgment, and the wisest of all mankind in his days. How great a philosopher he was, may be gathered from his writings; and amongst them are some indications that he was acquainted with the circulation of the blood, (not rightly described to us before Dr. Harvey wrote of it in 1628) the composition and frame of the human bodies, as well as others. His skill in architecture is sufficiently evident from the account we have of his building that most magnificent, beautiful, and perfect pile, the temple, and its furniture: but I cannot learn from the Divine writings, or from Josephus's fabulous history, how persons can form an idea in what orders, and in what manner exactly the workmanship was performed, as to give us exact draughts and models thereof: whether of the Corinthian or Ionic orders, with their members and mouldings, such as we have had transmitted down to us from the Greeks and Romans, and copy at this day. Solomon, no doubt, would have rejoiced to have added to him one blessing which we have enjoyed, the acquaint-

ance of (the glory of the age) Sir Isaac Newton, who, we may say, has exceeded all men since Solomon's time. It may, perhaps, be thought a presumption too nice and curious, to examine any of Solomon's great works mathematically: but as nothing is intended, nor can be said to lessen his wisdom or stupendous work, I shall only suppose a question was taken from that description the inspired writer has given us of his Molten Sea, 1 Kings vii. 23. ' And he made a ' Molten Sea ten cubits wide from brim to brim, round in compass, ' and five cubits high, and a line of thirty cubits did compass it about.' It is evident it could not be a circle; for then the line that compassed it must have been more than 30 cubits, viz. 31·4159265359. Therefore we may suppose it to be elliptical, (or that the workmen were not very curious in their measures, or not skilled in geometry) and the transverse diameter ten cubits, and its periphery thirty. What then would be the least triangle that could circumscribe the same? and how analytically to investigate the solution?

  N. B. A cubit is equal to 21·888 inches.

Answered by *Mr.* Turner.

  Let Ba = 10 cubits = b; x = the conjugate axis De; the periphery = 30 cubits = d; therefore a quadrant BGD = 7·5.

  The gentlemen of the Weekly Oracle have given us a theorem, by which nearly to find the periphery of an ellipsis, viz. To twice the square root of the sum of the squares of the two principal diameters, add one-third of the conjugate diameter, and it will give the circumference within less than one-hundredth part of the whole. Hence $2\sqrt{(bb+xx)}$ + $\frac{1}{3}x = d$; the root x = 9·056 cubits = 198·22 inches.
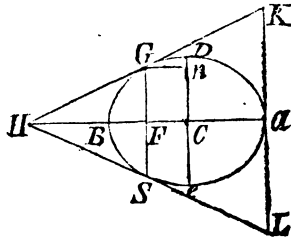
  But by fluxions, put BC = a = 5; DC = c; b = 7·5; x = Gn = Fc: Then, by the property of the ellipsis, F$G = y = \frac{c}{a} \sqrt{(aa-xx)}$, and $\sqrt{(\dot{x}^2 + \dot{y}^2)} = \frac{\dot{x}}{a} \sqrt{\frac{a^4 - aaxx + ccxx}{aa - xx}}$. Throw this into an infinite series, and the fluent will be $x + \frac{c^2 x^3}{6a^4} + \frac{c^2 x^5}{10a^6} - \frac{c^4 x^5}{40a^8} + \frac{c^2 x^7}{14a^8} - \frac{c^4 x^7}{28a^{10}} + \frac{c^6 x^7}{112a^{12}} + \frac{c^2 x^9}{18a^{10}}$, &c. which will be equal to the length of the arch Gd; and supposing x to flow till it becomes $x = a$,

then $a + \dfrac{c^2 a^3}{6a^4} + \dfrac{c^4 a^5}{10a^6} - \dfrac{c^4 a^5}{40a^8}$, &c. $= b = a + \dfrac{c^2}{6a} + \dfrac{c^4}{10a} - \dfrac{c^4}{40a^3}$, &c.

Now I find the powers of c above c^4 are very small, and may be set aside at present, and the law of the progression in c^2 being visible, we have $a + \dfrac{cc}{6a} + \dfrac{cc}{10a} + \dfrac{cc}{14a} + \dfrac{cc}{18a} + \dfrac{cc}{22a} + \dfrac{cc}{26a}$, &c. $= b$.

And by reduction $c^2 = 21 \cdot 17$, from which subtract the negative powers of c^4 which are equal to about $\cdot 52$. And then $c = \surd(20 \cdot 05) = 4 \cdot 544$, and the conjugate $= 9 \cdot 088$ cubits, nearly the same as before,

For the least circumscribing triangle.

Put $b = \text{B}a = 10$; $c = \text{D}e = 9 \cdot 056$; $z = \text{B}\text{F}$; $a\text{F} = b - z$; then the subtangent $\text{HF} = \dfrac{bz - z^2}{\frac{1}{2}b - z}$; and by the property of the ellipsis,

$b^2 : c^2 :: bz - zz : \text{GF}^2$; therefore $2\text{GF} = \sqrt{\dfrac{4ccbz - 4cczz}{bb}}$.

Now the $\triangle$ s HGS, HKL are similar, whence HF : (GS) $\sqrt{\dfrac{4ccbz - 4cczz}{bb}}$

$:: (\text{H}a) \dfrac{bb - bz}{b - 2z} : (\text{KL}) = \dfrac{b}{2z} \times \sqrt{\dfrac{4ccbz - 4cczz}{bb}}$; but $\frac{1}{2}$ KL $\times$ Ha

$=$ the area of the $\triangle$ HKL which is to be a minimum, i. e. $\dfrac{b}{4z} \times$

$\dfrac{bb - bz}{b - 2z} \sqrt{\dfrac{4ccbz - 4cczz}{bb}} =$ area; which reduced is $= \dfrac{b - z}{b - 2z}$

$\sqrt{\dfrac{b - z}{z}} =$ minimum; this in fluxions is $\dfrac{b\dot{z}}{(b - 2z^2)} \sqrt{\dfrac{b - z}{z}} -$

$\dfrac{b - z}{b - 2z} \times \dfrac{b\dot{z}}{2z^2 \sqrt{\dfrac{b - z}{z}}} = 0$; which equation, reduced, gives

$z = \frac{1}{4}b = 2 \cdot 5$. From whence the area of the $\triangle$ HKL is $= 117 \cdot 6$ cubits $= 39 \cdot 135$ square feet.

Additional Solution.

Put $a = 10 =$ transverse, $z =$ conjugate, and $c = 30 =$ circumference. Then, by Rule VI. p. 233 Mensuration, $\dfrac{a + z}{4} \times p$

$+ \frac{1}{2}p \sqrt{\dfrac{aa + zz}{2}} = c$, where p is $= 3 \cdot 1416$; hence $z = a - \dfrac{4c}{p}$

$$+ 4\sqrt{\left(\left(\frac{2c}{p} - a\right) \times \frac{c}{p}\right)} = 9.0876 = \text{the conjugate axe very true.}$$

Again, for the least triangle; since CF : FB :: aF : FH by the nature of the ellipse, and aF=FH in all curves, ∴ CF = FB = ¼aB. Then

$$a_H = 2a_F = \tfrac{3}{4}a_B, \text{ and } a_K = 2_{FG} = \frac{2_{DC}}{a_B} \sqrt{(a_F \times F_B)} = \frac{2_{DC}}{a_B} \times$$

$$\sqrt{(\tfrac{3}{4}a_B \times \tfrac{1}{4}a_B)} = \frac{_{DC}\sqrt{3}}{2}; \text{ consequently } H a \times a_K = \tfrac{3}{4}_{DC} \times a_B \times$$

$\sqrt{3} = 118.05 = $ the least triangle HKL required.

The proposer also solved this question, but the solution is omitted as it was so very falsely printed.   H.

IV. QUESTION 190, *by Mr.* John Bulman.

There is a cask, supposed the frustum of a parabolic conoid, or a cask of the 2d variety. The bung diameter is 38·4 inches; the head diameters are unequal, the greater 33·5, the lesser 28·8; the length of the cask 54·27. Required the content of the cask in ale gallons, the distance of each head diameter from the bung; it being supposed that there had been a decay in the cask, and cut off, and a new and larger head put in at one end; and to let us know the diameter, length, and content of the greatest cylinder that can be inscribed therein, the circumference of each base touching that of the cask around?

Answered by the Proposer.

Put $x = $ DC $=$ the difference of the distances of the two heads from the bung; $a=$NH$=19.2$; $b=$ BC $= 16.75$; $c = $ EF $= 14.4$; $d = $ CF $= 54.27$; then $.5d + .5x = $ DH $= $ HF ; and $.5d - .5x = $ CH : Per property of the curve $aa - cc : .5x + .5d :: aa : \dfrac{.5aax + .5aad}{aa - cc} = $ AH,

and $\dfrac{aax + .5ccd - .5ccx}{aa - cc} = $ AC, and
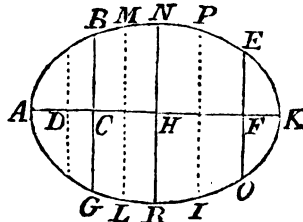
per prop. $\dfrac{aax + .5ccd - .5ccx}{aa - cc} : bb :: \dfrac{.5aax + .5aad}{aa - cc} : aa$; then, per 16 Eucl. 6, $a'x + .5aaccd - .5aaccx = .5aabbx + .5aabbd$; hence $x = \dfrac{bbd - ccd}{2aa - bb - cc} = $ DC $= 15.93174$. Hence the lesser head from the bung $= $ FH $= 35.10087$ inches, and CH $= 19.1691$.

The content of $\left\{\begin{array}{c}\text{NEOR}\\\text{NBGR}\end{array}\right\} = \left\{\begin{array}{c}112\cdot6191\\69\cdot3193\end{array}\right\}$ ale gallons, or $\left\{\begin{array}{c}137\cdot483\\84\cdot623\end{array}\right\}$ wine gallons.

The whole content 181·9384 ale gallons, or 222·106 wine gallons. The diameter of the greatest cylinder inscribed = BG = 33·5, length 38·23, and its content 119·8304 ale gallons, or 146·2842 wine gallons.

v. QUESTION 191, *by Mr*. J. Turner.

There is a stone in the form of a parallelopipedon, whose depth, breadth, and length are in arithmetical progression : the solid content is 5184 feet, and the common difference of the dimensions is 1-48th part of the product of the length and depth. Required the dimensions of the stone?

Answered by Mr. Dunthorne.

Put x, y, and z = the depth, breadth, and length of the stone, $b = 5184$, and $c = 48$: then $\dfrac{b}{y} = xz$, and $\dfrac{b}{cy} = \dfrac{xz}{c}$ = the common difference per question. Whence $y - \dfrac{b}{cy} = x$; and $y + \dfrac{b}{cy} = z$; consequently $y^3 - \dfrac{bb}{ccy} = b$; reduced $y^4 - by = \dfrac{bb}{cc}$. Solved, $y = 18$; whence $x = 12$, and $z = 24$.

vi. QUESTION 192, *by Mr*. C. Mason.

Find three such cube numbers, whose sum may be both a square and cube number; and if that sum be squared, to be a cube; also if cubed, shall be a square?

Answered by Mr. J. Hill.

The three numbers are $\dfrac{x^6}{8}$, $\dfrac{8x^6}{27}$, and $\dfrac{125x^6}{216}$; x being any number at pleasure. Hence if $x = 216$, we have three whole numbers for the answer.

Solution.

Let a^3, b^3, and c^3, be any three cubes whose sum is a rational cube; that is, let $a^3 + b^3 + c^3 = s^3$, then will as^2, bs^2, and cs^2 be the roots of three cubes whose sum is a 12th power, for $a^3s^9 + b^3s^9 + $

$c^3 s^9 = s^9 (a^3 + b^3 + c^3) = s^9 \times s^3 = s^{12}$; and therefore as^3x^2, bs^3x^2, cs^3x^2 will obviously be the roots of three cubes whose sum is a sixth power.

Now $3^3 + 4^3 + 5^3 = 6^3$, therefore we may take $a = 3$, $b = 4$, $c = 5$, and $s = 6$, and hence the preceding expressions for the roots of the three cubes will become $3 \times 6^3x^2$, $4 \times 6^3x^2$, and $5 \times 6^3x^2$, in which x may be expounded by any rational number whatever.

Take $x = \frac{1}{6}$, and then we shall have 18, 24, and 30 for the roots of the three cubes.

Again, $1^3 + 6^3 + 8^3 = 9^3$, therefore we may take $a = 1$, $b = 6$, $c = 8$, and $s = 9$; and hence $1 \times 9^3x^2$, $6 \times 9^3x^2$, and $8 \times 9^3x^2$ will be the roots of the three cubes in which x may be expounded by any rational number.

Take $x = \frac{1}{3}$, and the three roots will be 9, 54, and 72.

It will be perceived that the question admits of an infinite number of answers, but those brought out above are perhaps the lowest.   L.

THE PRIZE QUESTION, *by Mr.* Tho. Simpson.

> Young Strephon, long bless'd with his charming fair,
> In happy consort liv'd devoid of care;
> Till cruel fate call'd the fair nymph away
> From his kind arms, to cross the raging sea;
> Where horrid tempests, in thick darkness, roar,
> And toss'd his dearest to an unknown shore.
> He mourns, is restless, wanders day and night,
> In ev'ry clime to find his dear delight.
> Her lovely aspect his wing'd soul inspires;
> He longs, sighs, wishes, melts in soft desires.
> Propitious Venus pities his sad moan,
> Glides shining down from her etherial throne,
> And smiling says, (in a majestic tone)
>   ' Thrice ten degrees north of th' equator lies
>   ' This place, from which, as Sol due east doth rise,
>   ' Set out; and keep him always in your face;
>   ' Move not too fast, but such an equal pace,
>   ' To be eight miles more south at four hours end,
>   ' And you'll arrive to th' arms of your dear friend.'
> Thus said, she vanish'd from his wond'ring sight:
> But still the swain is in a mournful plight,
> Unless, fair nymphs, you'll vouchsafe to explain,
> And shew the place, that must the fair regain.

    i. e. the distance required to move in an hour.

Answered by Mr. Tho. Simpson, *the Proposer.*

Let the radius be $= 1$; then the sine of the latitude $= \frac{1}{2}$; for the versed sine of the hour from 6 put x; then the sine of the sun's azimuth will be $= \dfrac{1 - x}{\sqrt{(1 - \frac{3}{2}x + \frac{3}{4}xx)}}$;

now whilst x flows $\dot{x}$ the arch of time flows $\dfrac{\dot{x}}{\sqrt{(2x - xx)}}$;

which (because the man's motion is equal) must be in a constant ratio to the lineola $(rv) =$ the distance moved by the man in the same time : therefore by put-

ting (v) for the said ratio, we have $(rv) = \dfrac{v\dot{x}}{\sqrt{(2x - xx)}}$; and be-

cause the angle (mrv) is the complement of the sun's azimuth, it is as radius : sine of $(mrv) :: (vr) : (mv) = \dfrac{v\dot{x}}{2\sqrt{(1 - \frac{3x}{2} + \frac{3x^2}{4})}}$: the

fluent of which is $= \frac{1}{2} \times\; : x + \dfrac{3x^2}{8} + \dfrac{5x^3}{32} + \dfrac{27x^4}{512}$, &c. $\times\; v$, and when x becomes $\frac{1}{2}$, is the distance given $= \cdot308v = 8$, therefore $v = \dfrac{8}{\cdot308}$. But the fluent of (vr) is the arch of 4 hours; $60° \times v = 1\cdot0472v = 27\cdot2 =$ the distance moved in 4 hours; therefore 6·8 miles is the required distance.

The same answered by Merones.

Let AB be the line he describes, put $a =$ the space moved in an hour, $r =$ radius $= 1$; $s =$ sine, $c =$ cosine, $t =$ tangent, $f =$ secant, $q =$ arch of $60°$; AB $= z$, AK $= x$; $y =$ the sine of the hour from 6; $u =$ hour arch from 6 : then will $\dfrac{ry}{\sqrt{(rr - yy)}} =$ tan-

gent of hour. Let $\dot{z}$ and $\dot{u}$ be passed over in the same small part of time, then by uniform motion $q : 4a ::$
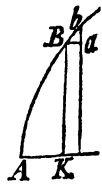
$\dot{u} : \dot{z} = \dfrac{4a\dot{u}}{q}$; and by the nature of the circle $\sqrt{(rr - yy)} : r :: \dot{y} :$

$\dfrac{r\dot{y}}{\sqrt{(rr - yy)}} = \dot{u}$. And by trigonometry $r : s :: \dfrac{ry}{\sqrt{(rr - yy)}}$:

$\dfrac{sy}{\sqrt{(rr - yy)}} =$ tang. azim. and $\dfrac{ty}{\sqrt{(ff - yy)}} =$ s. azim. from 6.

The angle $\text{B}ba = \odot$ azimuth from 6. And $r : \dfrac{ty}{\sqrt{(ff - yy)}} ::$

$(\text{B}b) \; \dot{z} : (\text{B}a) \; \dot{x} = \dfrac{ty\dot{z}}{r\sqrt{(ff - yy)}},$ &c. whence $a = \dfrac{qx}{4t \times 5829}$

$= \dfrac{8q}{4t \times 5329} = 6\cdot8072$ miles required.

———••●◐●●•———

Questions proposed in 1737, and answered in 1738.

I. QUESTION 193, by Mr. Merones.

Ye bright sons of art, that a rule did impart
    In the * Diary 1735, to obtain
The solid content of a conic segment,
    Cut off by a vertical plane. †
Since ye're so expert, proceed in like sort,
    \To compute us the surface convex
Of a segment the lesser, and all such to measure
    A general theorem annex.

Answered by the Proposer.

Through the axis $\text{B}n$, and any two points infinitely near each other as r, s, draw two planes $\text{B}rn$, $\text{B}sn$, cutting the base in the lines rn, sn, and the plane of the hyperbola in the lines ot, qi, parallel to the axis $\text{B}n$.

The ratio of the $\triangle$s nti, $\text{B}oq$, to each other, is compounded of the ratios of $\triangle nti$ to $\triangle nrs$, and $\triangle nrs$ to $\triangle \text{B}rs$, and of the $\triangle \text{B}rs$ to $\triangle \text{B}oq$, that is, of the ratio of nt^2 to nr^2, and of nr to $\text{B}r$, and of $(\text{B}r^2$ to $\text{B}o^2$, or of$)$ nr^2 to nt^2. Whence $\triangle nti : \triangle \text{B}oq$ $:: nr : \text{B}r$. And since $\triangle nrs :$ $\triangle \text{B}rs :: nr : \text{B}r$. Therefore the $\triangle nrs - \triangle nti : \triangle \text{B}rs - \triangle \text{B}oq$ $:: nr : \text{B}r$. That is, area $trsi :$ area $orsq :: nr : \text{B}r$. And since
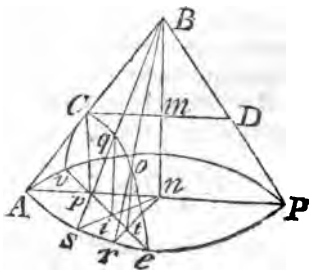
* Questoin 177, proposed in the Diary 1734, and answered in 1735.
† Perpendicular to the base.

this always holds for any small correspondent parts of the circle and the conic surface, it follows that the wholes are in the same ratio. Hence this theorem,

As the radius of the base : side of the cone :: segment of the base : surface of the segment of the cone.

And the area of the segment $eAvce$ will be found to be 904·38 square inches. The answer.

Mr. Ri. Dunthorne's answer to the same.

The curve surfaces of cones may be considered as made up of an infinite number of infinitely narrow annuli, whose radii are in arithmetical progression, and the cosines of their segments equal.

Let $r =$ radius, $d = 3\cdot14159$ &c. and $b =$ the cosine of any indefinite segment, then by the arithmetic of infinites, $dr - 2b -$
$$\frac{b^3}{3rr} - \frac{3b^5}{20r^4} - \frac{5b^7}{56r^6} - \frac{35b^9}{576r^8} - \&c. = \text{the arch of that segment.}$$

Put $R = An$, $B = cm$, $A = Ap$, $c = AC$; and let $e =$ the infinitely small breadth of one annulus. Then will $dre - 2Be -$
$$\frac{B^3e}{3rr} - \frac{3B^5e}{20r^4} - \frac{5B^7e}{56r^6} - \frac{35B^9e}{576r^8} - \&c. = \text{superficies of any indefi-}$$

nite annular segment, constituting the curve surface of such conic segment.

Therefore such curve surface will be composed of an infinite number of such series, having B constant, and r increasing in arithmetic progression, from B to its greatest R, and $\dfrac{c}{e} =$ the number of terms. Con-

sequently $\dfrac{dR^2c}{2A} - \dfrac{dB^2c}{2A} - 2BC + \dfrac{B^3c}{3RA} - \dfrac{B^3c}{3BA} + \dfrac{B^5c}{20R^3A} - \dfrac{B^5c}{20B^3A}$

$+ \dfrac{B^7c}{56R^5A} - \dfrac{B^7c}{56B^5A} + \dfrac{5B^9c}{576R^7A} - \dfrac{5B^9c}{576B^7A} + \&c. = \text{the curve surface}$

of such conic segment, which from the numbers in question 177, gives 905 square inches, the answer nearly the same as the proposer's above.

If we suppose $r =$ radius, and $a =$ versed sine of any indefinite segment of a circle; we shall, by the arithmetic of infinites, have

$$2\tfrac{1}{2} + \frac{4B^{\frac{1}{2}}A^{\frac{1}{2}}c}{3} + \frac{7A^{\frac{3}{2}}c}{15B} - \frac{71A^{\frac{5}{2}}c}{840B^{\frac{3}{2}}} + \frac{319A^{\frac{7}{2}}c}{10080B^{\frac{5}{2}}} - \frac{5419A^{\frac{9}{2}}c}{354816B^{\frac{7}{2}}} + \&c.$$

$=$ the curve superficies of such conic segment, in numbers above.

II. QUESTION 194, by Mr. J. Turner.

An elliptical field, whose principal axes are 40 and 60 chains, is, by a father's will, to be divided between his two sons in the following manner. A semi-ordinate is to be drawn at right angles to the transverse axis at the distance of 15 chains from one end thereof,

and the parting fence to be the shortest line possible bisecting the
semi-ordinate. Required the shares and the length of the parting
fence?

<center>*Answered by the author* J. T.</center>

Let CD be the line sought. Put AS $= 30 = t$, MS $= 20 = n$,
AE $= 15 = a$, GE $= 8\cdot66 = b$; EF $= z$,
SB $= x$, AB $= t - x$, BC $= y$, FB $= t - x$
$- a + z = m - x + z$.

As $b : z :: y : \dfrac{yz}{b} =$ FB. And, by
the property of the ellipse, $tt : nn ::$
$tt - xx : \dfrac{ttnn - nnxx}{tt} = yy$. Ergo $x =$
$\sqrt{\dfrac{ttnn - ttyy}{nn}}$. Put $r = tt$, $s = \dfrac{tt}{nn}$.
Then $x = \sqrt{(r - syy)}$. Or, putting for
x its value, and transposing, $\dfrac{mb + bz - yz}{b}$
$= \sqrt{(r - syy)}$; and, squaring the parts,
$mmbb + 2mbbz - 2mbzy + bbzz - 2bzzy + zzyy = bbr - bbsyy$.
Put $q = bbs + zz$, $2p = 2mbz + 2bzz$; $g = bbr - mmbb - bbzz$
$- 2mbbz$. Then we have $qyy - 2py = g$. And extracting the root,
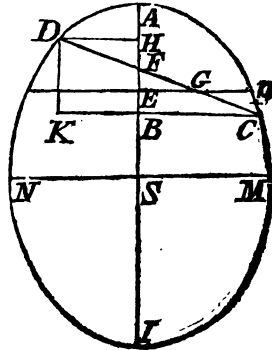
$y = \dfrac{p}{q} \mp \sqrt{\left(\dfrac{g}{q} + \dfrac{pp}{qq}\right)}$. Hence it is evident that $\sqrt{\left(\dfrac{4g}{q} + \dfrac{4pp}{qq}\right)}$

is equal to the difference of the double value of y; that is, of the
lines $+$ CB and $-$ DH, or CK. Lastly, GE^2 : GF^2 :: CK^2 : CD^2 which
is to be a maximum; which in symbols will be, when the value of
q, p, and g are restored,

$= \dfrac{bb + zz + 4b^4rs - 4mmb^4s - 4b^4szz - 8mb^4sz + 4bbrzz}{b^6ss + 2b^4szz + bbz^4}$. And,

dividing by bb, and substituting letters for the known quantities,
$\dfrac{h - kz + lz^2 - fz^3 + \varphi z^4}{a + \delta zz + z^4}$, in fluxions, $= 0$. It is evident that this
will only produce an equation of the sixth power, by which the value
of z comes out $=$ EF.

Mr. Lord finds the length of the fence 32·03 chains : One share
161 a. 14 p. The other 27 a. 1 r. 20 p.

Merones, by a curious and short process, finds CD the fence $=$
33·84 ; the greater segment 1561·45 ; the lesser 323·511.

From a geometrical construction I find the numbers nearly as
follow : EF $= 2\cdot65$ chains; GE $= 8\cdot6$; SB $= 12\cdot12$: BE $= 2\cdot88$;
CD the shortest fence 33·25; CB $= 18\cdot15$; EP $= 17\cdot2$; AB $= 18\cdot0$;
FB $= 5\cdot38$. And the content of the segment DA*p*CD $= 28$ a. 3 r.

38 p. The greater CMINDC = 159 2 1, the shares; and the whole 188 1 39.

<p style="text-align:center;">III. QUESTION 195, <i>by Mr.</i> Rich. Lycett.</p>

There is an oak tree (the frustum of a cone) whose length is ten yards, the diameter at the top one foot, at the bottom three feet; and an ivy twisteth round it in the manner of a spiral screw, so that each twist is ten inches distant; three-sevenths of the ivy is eat into the tree; the diameter of the ivy is at the bottom one foot, at the top three inches. Quere the length of the ivy, and the content of both?

<p style="text-align:center;"><i>Answered by the Proposer.</i></p>

The centre of the ivy will be found to lie without the periphery of the oak at the bottom ·055, and at top ·015 parts of a foot; which doubled and added to the diameters, at the bottom 3·11, and at top 1·03. Then the frustum's length on the outside is 30·91, and the angle with the base is 88° 0′ 52″. The circumference at bottom 9·77, and at top 3·235 feet. Then a line being drawn through the extremity of the top diameter and parallel to the central line is a mean arith. proportional; which multiplied by the number of twists 36·02 gives 234·0, which let be one leg of a triangle, 30·01 the other, with the angle between them 88° 0′ 52″. The other side will be found 235·3 feet the length of the ivy: Hence the content of the ivy is 80·87 feet, three-sevenths of which is 34·6: the content of the tree with the ivy growing into it is = 102·1; consequently 102·1 — 34·6 = 67·5 feet, the real content of the tree.

<p style="text-align:center;">Merones's <i>Answer to the same.</i></p>

In the cone compleated, let s = the length of the side, c = the circumference at the base, d = the distance of the spiral threads, z = the spiral line, and x = the distance from the vertex to the top of the first helix or turn.

Then $d : c :: \dot{x} : \dfrac{c\dot{x}}{d}$, and $s : \dfrac{c\dot{x}}{d} :: x : \dfrac{cx\dot{x}}{ds}$ = a small space the ivy moves round. And let the hypothenuse line the ivy moves round in be = $\dot{z}$; then $\dot{z} = \sqrt{\left(\dot{x}^2 + \dfrac{cc x^2 \dot{x}\dot{x}}{d^2 s^2}\right)} = \dfrac{c\dot{x}}{ds}\sqrt{\left(\dfrac{ddss}{cc} + xx\right)}$. And, finding the fluent, $z = 237\cdot746$ the length of the spiral ivy. Hence the content of the oak = 102·102; of the ivy = 81·6925 feet.

Remark. The fluent of $\dfrac{c\dot{x}}{ds}\sqrt{\left(\dfrac{ddss}{cc} + xx\right)}$ is = to $\dfrac{ds}{2c}\varrho + \dfrac{cx}{2ds}$

$\times \sqrt{\left(\dfrac{ddss}{cc} + xx\right)}$, by Emerson's 13th form, where ϱ is =2·30258×

log. $x + \sqrt{\left(\dfrac{ddss}{cc} + xx\right)}$, the fluent of $\dfrac{\dot{x}}{\sqrt{\left(\dfrac{ddss}{cc} + xx\right)}}$ by the 9th

form. L.

IV. QUESTION 196, *by Mr.* Robert Heath.

Three ships, A, B, and C, sailed from a certain port in north lati-
tude, until they arrived at three different ports, all lying under the
equinoctial; A sailed on a direct course, between the south and the
west 175·62 leagues; C sailed 133 leagues between the south and
east; B sailed a course between A and C 102 leagues, making the an-
gle or rhumb with A, equal the angle that C made with the equinoc-
tial. Hence it is required to find the port sailed from, each ship's
course, and distance from each other, and their respective ports?
and to solve it by an equation not higher than a quadratic?

Answered by the Proposer.

As 102 : 175·62 :: 133 : 228·99 leagues : The port sailed from
is in 5° 5′ north latitude.

The course of $\left\{\begin{array}{l}\text{A } 54° \ 38' \text{ west.} \\ \text{B } \ \ 4 \ \ 47 \text{ west.} \\ \text{C } 40 \ \ 10 \text{ east.}\end{array}\right\}$ diff. long. $\left\{\begin{array}{l}·7° \ \ \ 9' \\ 0 \ \ 25 \\ 4 \ \ 17\end{array}\right\}$ Req.

Additional Solution.

The reason of the proportion, in the above original solution, for
finding the distance (228·99) between the
extreme ports, will clearly appear from
the annexed figure, in which P represents
the port sailed from, and A, B, C the
ports arrived at by the respective ships.
For, since the $\angle c = \angle$ APB by the question,
and the $\angle$ A common to the two △s APB,
APC, therefore the third angles are equal,
and those triangles equiangular. Hence then PB (102) : PA (175·62)
:: PC (133) : AC = 228·99. And in like manner as AC : AP :: AP :
AB. Hence the perpendicular and the angles are easily found.
Hence also the problem will be easily constructed. For it is only
taking AC a fourth proportional to the three given lines. H,
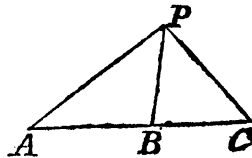
V. QUESTION 197, *by Mr.* Ant. Thacker.

Let AB = 1000, BC = 2000, CD = 3000, DE = 4000, and AE

variable; shew how to find the greatest area that can possibly be
included by these five lines.

Answered by Mr. Rob. Heath.

Let AB $= 1000 = b$, BC $= 2000 = c$, CD $= 3000 = d$, DE $=$
$4000 = f$; and let $a =$ AE the diameter,
radius $= 1$.
Then as

$$a : 1 :: b : \frac{b}{a} = \text{s. } \angle \text{BEA};$$

$$a : 1 :: f : \frac{f}{a} = \text{s. } \angle \text{DAE}.$$

Now as $b : \dfrac{b}{a} :: c : \dfrac{c}{a} = \text{s. } \angle \text{BAC} =$
$\angle$ BEC by the nature of a circle.

And as $f : \dfrac{f}{a} :: d : \dfrac{d}{a} = \text{s. } \angle \text{CED} = \angle \text{CAD}.$
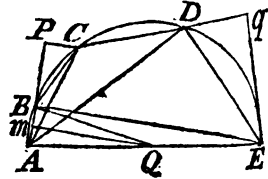
The $\angle$ BEA $+ \angle$ CEB $= \angle$ AEC whose sine $= \dfrac{b}{a} \sqrt{\dfrac{aa - cc}{aa}} +$
$\dfrac{c}{a} \sqrt{\dfrac{aa - bb}{aa}}$; and the $\angle$ CAD $+ \angle$ DAE $= \angle$ CAE whose sine $=$
$\dfrac{d}{a} \sqrt{\dfrac{aa - ff}{aa}} + \dfrac{f}{a} \sqrt{\dfrac{aa - dd}{aa}}$; then say as $1 : a :: \dfrac{b}{a} \sqrt{\dfrac{aa - cc}{aa}} +$
$\dfrac{c}{a} \sqrt{\dfrac{aa - bb}{aa}} : b \sqrt{\dfrac{aa - cc}{aa}} + c \sqrt{\dfrac{aa - bb}{aa}} = $ AC. And

as $1 : a :: \dfrac{d}{a} \sqrt{\dfrac{aa - ff}{aa}} + \dfrac{f}{a} \sqrt{\dfrac{aa - dd}{aa}} : d \sqrt{\dfrac{aa - ff}{aa}} +$
$f \sqrt{\dfrac{aa - dd}{aa}} = $ CE. But AC$^2 +$ CE$^2 =$ AE^2 by 31 Eucl. III. which
is in algebraic terms,

$$\left. \begin{array}{l} dd - \dfrac{2ddff}{aa} + ff + 2df \sqrt{\dfrac{a^4 - a^2 dd - a^2 ff + ddff}{a^4}} \\[4mm] bb - \dfrac{2bbcc}{aa} + cc + 2bc \sqrt{\dfrac{a^4 - a^2 bb - a^2 cc + bbcc}{a^4}} \end{array} \right\} = aa.$$

Whence by reduction and the converging series a is found $=$
6646·316, &c.

But the diameter AE or radius AQ may be more expeditiously found
by a table of natural sines (which is an invention of my own) by
finding out four chords, in the proportion of the chords given AB, BC,
CD, DE. By a few trials, I find the angle AQm under half the chord

$= 8° 39'$ the next less to a minute; and $8° 40'$ the next greater. The operation is thus :

	o	/	//	///	Nat. sines						
Take	8	39	0	0=	1503·981	8	40	0	0=		
angles	17	30	19	33 = 2 ×	1503·981	17	32	23	57=2×		
corre-	26	49	13	11 = 3 ×	1503·981	26	52	32	40=3×		
spond.	36	59	2	30 = 4 ×	1503·981	37	3	59	42=4×		
	89	57	35	14		90	8	56	19		
Comp.	0	2	24	46 diff.		Sub. 89	57	35	14 diff.11 21 5		

(rotated: 1506·857 = Nat. sines)

Say, as $11' 21'' 5'''$ to $60''$, so is $2' 24'' 46'''$ to $12'' 45'''$ to be added to the next lesser angle : Whence $8° 39' 12'' 45'''$ is the true angle AQm; consequently $17° 18' 25'' 30''' =$ angle AQB; and since AQB is an isosceles triangle, the angle QAB $=$ ABQ $= 81° 20' 47'' 15'''$. By trigonometry, say $17° 18'$ &c. ($2974·929$ N. S.) : 1000 :: $81° 20'$ &c. ($9886·161$) : $3323·158 =$ AQ.   Whence AE $= 6646·316$ the diameter required.

Consequently the greatest area of the polygon · ABCDEA is $= 14567·640$ &c. very near the truth.

The same answered by Merones.

In the last figure produce the lines AB, CD, and let fall perpendiculars on them from C and E.

When the area is the greatest possible, it is plain the angles ACE, ABE, and ADE must be right angles, therefore the figure ABCDE is to be inscribed in a semicircle whose diameter is AE.   Let AB be called a, BC $= b$, CD $= c$, DE $= d$, BP $= x$, AE $= u$.   The $\triangle$s ACE, BCP, DEq are similar; therefore $b : x :: u : \dfrac{xu}{b} =$ EC.   And $b :$

$\sqrt{(bb - xx)} :: d : \dfrac{d}{b} \sqrt{(bb - xx)} =$ Dq.   And by Eucl. II. 12.

$\dfrac{xxuu}{bb} = cc + dd + \dfrac{2cd}{b} \sqrt{(bb - xx)} =$ CE².   And $aa + bb + 2ax$

$=$ AC²   And by Eucl. I. 47, AC² $+$ CE² $=$ AE².   Or $uu = aa + bb$

$+ cc + dd + 2ax + \dfrac{2cd}{b} \sqrt{(bb - xx)} = ss + 2ax + \dfrac{2cd}{b}\sqrt{(bb-xx)}$

putting ss for the known quantities.   Hence we have $ss + 2ax + \dfrac{2cd}{b} \sqrt{(bb - xx)} = \dfrac{bbcc + bbdd + 2bcd\sqrt{(bb - xx)}}{xx}$.   Which reduced

gives $x = 1795·0355$; and AE $= 6646·31856$; and the greatest area $14567·718\frac{1}{4}$.

VI. QUESTION 198, *by Mr.* Christ. Mason.

There is a high cliff from the top of which a heavy body is suffered to fall, the time of descent is 11 times as much as the ascent of sound from the bottom to the top. What is the height of the cliff?

Answered by Mr. Mason.

Let a be the time a heavy body requires in falling from the top of the cliff; then $a \div 11 =$ the time that sound requires to move that space; $b = 16$ feet; $c = 968$ feet.

Then $baa = ca \div 11$ per quest. and $11baa = ca$, which divide by a, and it is $11ba = c$; therefore $a = c \div 11b = 5\cdot5$ seconds, the time sought. Then $5\cdot5 \times 5\cdot5 \times b = 484$ feet, the height sought. But if $b = 16\frac{1}{12}$ the answer is $481\cdot417$.

THE PRIZE QUESTION, *by Mr.* Turner.

In the triangle BAD, BD $= 55$, DA $= 53$, and BA $= 40$ feet; about A as a centre a circle is described with a radius of 15 feet; it is required to determine a point F, in the periphery of this circle to which if right lines BF and DF are drawn the angle BFD may be the greatest possible?

Answered by Mr. Rob. Heath.

In the triangle ABD, there is given AB $= 40 = b$, AD $= 53 = c$, BD $= 55 = 2d$, PS $= 30 = 2f$.

Make PS parallel to BD; draw mc perpendicular. On the centre c describe a circle, to touch the given one at F, through the points B, D; the point of contact F, from which lines must be drawn to B and D to constitute the greatest angle, which is a maximum. Draw CA, CD, and Ap the perpendicular of the triangle. In order to investigate the centre c, and the lines BF and DF, find the perpendicular A$p =$
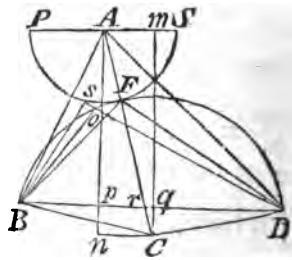
$86\cdot4342 = mq = h$; $qp = $ A$m = 10\cdot991 = g$; and let $qc = a$.

Then per 47 Eucl. I. $\sqrt{(aa + dd)} = $ CD $= $ CF; $\sqrt{(aa + dd)} + f = $ AC; and $\sqrt{(gg + hh + 2ha + aa)} = $ AC $= \sqrt{(aa + dd)} + f$. Whence $gg + hh + dd - ff + 2ha = 2f\sqrt{(aa + dd)}$ by involution and transposition, or $k + 2ha = 2f\sqrt{(aa + dd)}$ by substitution.

Whence $aa + \dfrac{kk}{hh - ff} a = \dfrac{4ffdd - kk}{4hh - 4ff}$. And $a = 5\cdot10647$; consequently AC $= 42\cdot97$. Now the $\triangle$s ACm and Apr are similar, mc : AC :: Ap : Ar $= 37\cdot687$; AC $-$ Ar $= rc = 5\cdot283$; whence Fr $=$ $22\cdot687$; Br $= 26\cdot139$; $\angle$cAm $= 75°\ 11'$; $\angle r$BF $= 47°\ 9\frac{1}{2}'$; BF $=$ $22\cdot91$; DF $= 41\cdot02$; $\angle$BFD $= 100°\ 30'$ a maximum.

The same answered by the Proposer.

In the last figure draw An perpendicular to BD, and cn perpendicular to An, and cq perpendicular to BD. Put Bq $= 27\cdot5 = m$, AF $= 15 = b$, Ap $= 36\cdot4 = c$, $pq = nc = 11\cdot1 = d$; and CB $=$ CF $= x$, $cq = np = y$. Then is CA $= b + x$, and An $= c + y$.

And by 47 Eucl. I. $\begin{cases} xx = yy + mm \\ bb + 2bx + xx = cc + 2cy + yy + dd. \end{cases}$
Substitute for xx its value $yy + mm$, and then $bb + 2b\sqrt{(yy + mm)}$ $+ yy + mm = cc + 2cy + yy + dd.$ Put $n = cc + dd - bb -$ mm, then $2b\sqrt{(yy + mm)} = n + 2cy$; and, involving, $4bbyy +$ $4bbmm = nn + 4ncy + 4ccyy$. Lastly put $2p = \dfrac{4nc}{4cc - 4bb}$, and $q = \dfrac{4bbmm - nn}{4cc - 4bb}$; then $yy + 2py = q$. And extracting the root, $y = \sqrt{(pp + q)} - p = 5\cdot1$; from whence the angle BFD $=$ $100°\ 30'$.

Answered by Merones.

Draw the lines Ds, Bs, Bo; then the angle BFD or BOD $=$ BSD $+$ sBo; whence BFD is greater than BSD. To find the radius, let Bq $= b$, $pq = c$, Ap $= p$, AF $= r$, $cq = x$. Then will BC $=$ CF, or $\sqrt{(bb + xx)}$ $= \sqrt{(cc + pp + 2px + xx)} - r$. Thence $cc + pp + 2px = bb +$ $rr + 2r\sqrt{(bb + xx)}$. And (putting $ss = cc + pp - bb - rr$) $ss +$ $2px = 2r\sqrt{(bb + xx)}$. And lastly $\dfrac{4pp}{4rr}xx + 4pssx = 4bbrr - s^4.$ Whence $x = 6\cdot38716$, and FC $= 29\cdot19$, and the $\angle$BFD $= 100°\ 32'$. Q. E. I.

Additional Solution.

This problem is very easily constructed in all its cases, viz. when the angle is to be either a maximum, or a minimum, or a given quantity. For when it must be a given quantity, it is only describing on the given base BD a segment of a circle to contain the given angle, which segment will cut the given circle in two points, either of which

will do. When the angle is to be a maximum or a minimum, the circle BFD must touch the given circle PFS respectively below or above the vertex A; which is very well done in the 12th prob. of Lawson's Apollonius on Tangencies.   H.

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Questions proposed in 1738, and answered in 1739.

I.   QUESTION 199, by Mr. Tho. Bird.

A gentleman has a garden in the form of a rhombus, the side of which is $8\frac{3}{4}$ chains, and the side of the inscribed square is 6 chains: What is the area?

Answered by Mr. Rob. Heath.

Let $b =$ the sides AB $=$ BC, &c. $= 8\cdot75$ chains, $2c = ef = fh$, &c. $= 6$ chains; then it is evident $ez = zf = 3$ chains $= c$: Let BZ $= x$. By similar $\triangle$s, $x : c :: x + c :$ $\dfrac{x + c}{x}c =$ AO. But AO$^2 +$ BO$^2 =$ AB^2 by 47 Eucl. I. which is in algebraic terms $\dfrac{xx + 2xc + cc}{xx}cc + xx + 2xc$ $+ cc = bb$. By reduction $(xx+cc)^2$ $+ 2cx \times (xx + cc) = bbxx$: by comp. □ and extraction $xx + cc + xc$ $= x \sqrt{(bb + cc)} = 9\cdot25x$; whence the equation becomes $xx - 6\cdot25x =$ $- cc$. By comp. □ and extraction a second time, $x = \pm \sqrt{(9\cdot765625 -}$ $cc) + 3\cdot125 = 4$; or $2\cdot25$ (according to the ambiguity of the quest.) whence the area of the rhombus or garden ABCD $= 73\cdot5$ chains $= 7$ ac. 1 r. 16 p.
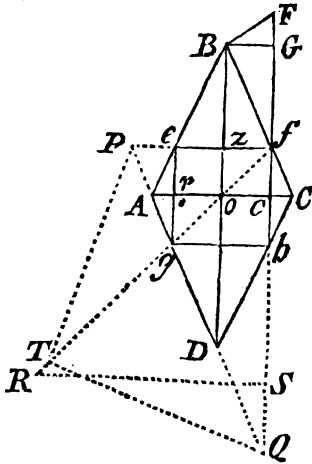
The same answered by Mr. Hen. Travis, of Wirksworth.

Erect the line BF perpendicular to CB, continue the line hf till it cuts BF at F, let fall the perpendicular BG, which is equal to $zf = a$;

call f_F, x; and f_B, y; then will $cf = b - y$: Now it is evident that the rectangled triangles, ccf, FBf, and FGB are similar, and that ccf and FGB are equal, because $cf = GB$; hence per 47 Eucl. I. $f_B^2 + BF^2 = f_F^2$; that is, $yy + bb - 2by + yy = xx = bb - 2by + 2yy$; and $fc : fc :: f_B : f_F$, or $a : b - y :: y : x$, $\therefore ax = by - yy$, and $-2ax = -2by + 2yy$. But $bb - 2by + 2yy = xx$. Now if for $-2by + 2yy$ we put $2ax$, its value found, we have $bb - 2ax = xx$, or $xx + 2ax = bb$. Hence $xx + 2ax + aa = bb + aa$; $\therefore x + a = \sqrt{(bb + aa)} = 9.25$; $ax = by - yy$, or $yy - by = -ax$, and $yy - 8.75y = -18.75$; then $yy - 8.75y + 4.375^2 = -18.75 + 4.375^2$; $y - 4.375 = \sqrt{(-18.75 + 4.375^2)}$; $\therefore y = 5$. And the area of the rhombus $= 7$ a. 1 r. 16 p.

Additional Solution.

Produce the side AD of the rhombus both ways to meet the opposite sides fe, fh, of the square, produced in P and Q; then will PQ be $= 2$AD: For, $ef = 2ro$, and $Pe = 2Ar$ (because $ge = 2gr$), $\therefore Pf = 2Ao$, and by sim. $\triangle$s PQ $= 2$AD. The problem is therefore reduced to this, To apply a given line PQ (or 2AD) between two lines fe, fh, given in position, and to pass through a given point g; which is one of the problems of Apollonius *Concerning Inclinations*, and which may be thus easily constructed.

Construction. In fg produced take $f_R = 2$AD the given line, and draw RS $\parallel gh$; take also gT such that the rectangle frg be $= RS^2$; then with the centre T and radius RS cross fe and fh in P and Q, which will be the points required.

Demonstration. Since $fTg = RS^2 = TP^2$, $\therefore Tf : TP :: TP : Tg$, and consequently the $\triangle$s TPf, TPg are similar by Eucl. VI. 6, $\therefore$ the $\angle$TPg $=$ the $\angle$TfP $= \angle$RfS or $\angle f$RS; in like manner the $\angle$TQP $= \angle$RfS; consequently the $\triangle$ TPQ is similar to the $\triangle f$RS; but RS or sf $=$ TP or TQ, $\therefore$ also PQ $= f_R = 2$AD.   H.

II. QUESTION 200, *by Mr.* Ri. Lovatt.

In the spherical triangle ACB, the angular points A, c, B, denote the places of three fixed stars, and the point D the zenith of the place of observation, and there is given AB $= 40°$, BC $= 60°$, AC $= 67° 28'$, $\angle$ADC $= 148° 6'$, $\angle$CDB $= 121° 54'$; Required AD, BD, and CD, the complements of the altitudes?
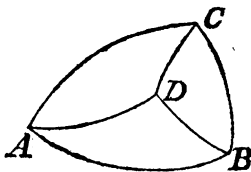
Answered by Mr. Robert Heath.

$$\text{Given sides}\begin{cases} \text{AC} = 67°28' = a \\ \text{BC} = 60 \quad 0 = b \\ \text{AB} = 40 \quad 0 = c \end{cases} \; \text{Nat. s.} \quad \bigg| \quad \text{Given angles} \begin{cases} \text{ADC} = 148° \; 6' = d; \\ \text{CDB} = 121 \; 54 = e; \\ \text{ADB} = 90 \quad 0 = 1. \end{cases}$$

Whence by trigonometry is found the angle CAB $= 69° \, 39'$, ABC $= 90° \, 00'$, ACB $44° \, 7'$. Let $x =$ BD. Then by spheric trigonom. $c : 1 :: x : x \div c = $ s. $\angle$DAB. Let $m =$ sine angle CAB, and n its cosine. Then $m \sqrt{\dfrac{cc - xx}{cc}} - \dfrac{nx}{c} = $ s. $\angle$CAD. Say $d : a ::$

$m \sqrt{\dfrac{cc - xx}{cc}} - \dfrac{nx}{c} : \dfrac{am}{dc} \sqrt{(cc - xx)} - \dfrac{nax}{cd} = $ DC. Now the sine

$\angle$ABD $= \sqrt{\dfrac{cc - xx}{cc - ccxx}}:$ Then since $\angle$ABC $=$ a right angle, there-

fore the complement of $\angle$ABD $= \dfrac{x}{c}\sqrt{\dfrac{1 - cc}{1 - xx}} = \angle$DBC. By

spherics, $e : b :: \dfrac{x}{c}\sqrt{\dfrac{1 - cc}{1 - xx}} : \dfrac{bx}{ce}\sqrt{\dfrac{1 - cc}{1 - xx}} = $ DC $= \dfrac{am}{dc}\sqrt{(cc - xx)}$

$- \dfrac{nax}{cd}$. Which equation reduced, and turned into numbers, will be

$$2.549528 \sqrt{(.4131758987 - xx)} - .945629x = \dfrac{1.215692852x}{\sqrt{(1 - xx)}}.$$

Solved, $x = .47477$, &c.

$$\text{Whence}\begin{cases} \text{DB} = 28° \; 21' \\ \text{CD} = 40 \quad 58 \\ \text{AD} = 29 \quad 29 \end{cases} \text{whose complements} \begin{cases} 61° \; 39' \\ 49 \quad 2 \\ 60 \quad 31 \end{cases} \text{are the alt. of the stars required.}$$

The Proposer Mr. Ri. Lovatt's *answer.*

Put $b =$ sine of AB $\div$ by sine of the $\angle$ADB : $p =$ cosine AC ; $d =$ sine BC $\div$ by s. $\angle$CDB ; s. ABC $= n$; and $m =$ its cosine ; $a =$ s. $\angle$ABD. Then $ab =$ s. alt. AD. And $m \sqrt{(1 - aa)} - am$ will be $=$ s. $\angle$DBC ; and $nd \sqrt{(1 - aa)} - amd =$ s. CD. Hence this analogy, as radius 1 : cosine AD $= \sqrt{(1 - aabb)} :: $ cosine CD $= \sqrt{(2anmd^2\sqrt{(1 - aa)}} - aamm + aann - nn + 1) : p$. Therefore $(1 - aabb) \times (2anmd^2 \sqrt{(1 - aa)} - aamm + aann - nn + 1) = pp$. Which gives AD $= 29° \, 29'$, BD $= 28° \, 21'$, CD $= 40° \, 50'$; whose complements are the altitudes.

III. QUESTION 201, *by Mr.* Christ. Mason.

Suppose a ship set sail from the latitude 51° north, and shape a north-west course, and sail without interruption ; Where will it at last arrive, and how many leagues run ?

Answered by Mr. J. May, jun. of Amsterdam.

The ship will arrive at the north pole, and the leagues run, will be the length of the loxodromic beginning at the latitude of 51 north, and ending at the pole, after having performed infinite revolutions about the said pole; the length of the said loxodromic will be 1278·56 leagues, supposing the earth to be a true spherical figure; and the circumference according to Norwood's observations, likewise the ship infinite small.

The same answered by Mr. Walter Trott, per Corollary to Prob. XII. Stone's Appendix.

Suppose a line drawn infinitely near to PB; then may the triangle made thereby be looked on as rectilinear; and making nm radius, mr will be the secant of the course, and $a = $ PA $=$ PB ; A$m = y$; and $mn = \dot{x} = $ fluxion of the diff. latitude; then per plain trigonometry, R : sec. course :: $\dot{x}$: $\dot{y}$. Hence, taking the fluents, $y = $ secant of the course multiplied by x, divided by the radius. The ship's distance sailed is 1103·08 leagues, and then coincides with the pole.
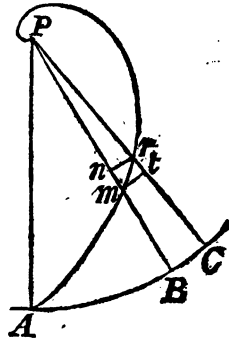
Mr. Tho. Bird answers thus:

The small decrements of the spiral, or the distance the ship runs while it passes through small, but equal angles of longitude (as at every 0·01 deg.) are a series of geometrical proportionals continued decreasing and ending in 0, in the pole: whose ratio (1·0001745) together with the first or greatest term (0·57736) and last or least (0) being known; the sum of all the series or length of the spiral is easily found $=$ 3309 minutes of a degree, or 1103 leagues; which may more easily be done by fluxions. However it may be solved to a geometrical exactness by this easy and known analogy.

As the cosine of the course ......co. ar. 0.150515
To diff. lat. 780 leagues........... 2·192095
So radius..................... 10·000000
To dist. run 1103 leagues .... 3·042610

After the ship has passed through all the degrees of longitude, and is arrived to the same meridian itself, it will be but 4·632 minutes from the pole, according to artificial tangents (for Wright's meridional parts being made from false secants, are not to be trusted) after

which it will be no deviation from mathematical exactness to suppose the remainder to the pole a plane, in which case the proportion of the velocity of the ship's approach towards the pole, is nearly as 10000000 to 18151, she in the next round will be but 0·00841 min. from the pole, and the next revolution falls into the pole itself, contrary to what the ingenious Oughtred supposed in his Cir. Prop. of Nav. p. 37.

Mr. Chr. Mason, *the Proposer, answers in this manner.*

Let P denote the north pole, A the place whence the ship set sail, AmP its course, AP the meridian, and BP, CP two other meridians infinitely near. Nr a parallel of latitude. Let AP = co. lat. = b; r = rad. c = cosine of the course, Bm = a; AmP = x = dist. run. Then by plain trigonometry, $a : \dfrac{ra}{c} :: b : \dfrac{bra}{ca} = \dfrac{br}{c} = x = 1277\cdot73$ leagues.

The same answered by Mr. R. Dunthorne.

Since parts of rhumbs are every where to their corresponding parts of the meridians, as radius to the cosine of the course, it will be S. 45 : rad. :: 900·25 leagues (the dist. of 51° from the pole) : 1273·14 leagues the answer.

Additional Solution, taken from the Third Volume of the Course.

PROBLEM. *Suppose a ship to sail from the Orkney Islands, in latitude 59° 3′ north, on a N.N.E. course, at the rate of 10 miles an hour; it is required to determine how long it will be before she arrives at the pole, the distance she will have sailed, and the difference of longitude she will have made when she arrives there?*

Let ABC represent part of the equator; P the pole; AmrP a loxodromic or rhumb line, or the path of the ship continued to the equator; PB, PC, any two meridians indefinitely near each other; nr, or mt the part of a parallel of latitude intercepted between them.

Put c for the cosine, and t for the tangent of the course, or angle nmr to the radius r; Am any variable part of the rhumb from the equator, = v; the latitude Bm = w; its sine x, and cosine y; and AB, the difference of longitude from A, = z. Then since the elementary triangle mnr may be considered as a right-angled plane triangle, it is, as rad. $r : c =$ sin. $\angle mrn :: v = mr : w = mn :: v$: w; therefore $cv = rw$, or $v = \dfrac{rw}{c} = \dfrac{sw}{r}$, by putting s for the secant of the $\angle nmr$ the ship's course. In like manner, if w be any

other latitude, and v its corresponding length of the rhumb; then v
$= \dfrac{rw}{c}$; and hence $v - v = r \times \dfrac{w - w}{c}$, or $D = \dfrac{rd}{c}$, by putting
$D = v - v$ the distance, and $d = w - w$ the difference of latitude;
which is the common rule.

The same is evident without fluxions: for since the $\angle mrn$ is the
same in whatever point of the path AmrP the point m is taken, each
indefinitely small particle of AmrP, must be to the corresponding in-
definitely small part Bm, in the constant ratio of radius to the cosine
of the course; and therefore the whole lines, or any corresponding
parts of them, must be in the same ratio also, as above determined.
In the same manner it is proved that radius : sine of the course ::
distance : the departure.

Again, as radius $r : t =$ tang. $nmr :: w = mn : nr$ or mt, and
as $r : y :: $ PB : P$m :: z =$ BC : mt; hence, as the extremes of these
proportions are the same, the rectangles of the means must be equal, viz.

$y\dot{z} = t\dot{w} = \dfrac{tr\dot{x}}{y}$ because $\dot{w} = \dfrac{r\dot{x}}{y}$ by the property of the circle;

therefore $\dot{z} = \dfrac{tr\dot{x}}{yy} = \dfrac{tr\dot{x}}{r^2 - x^2}$; the general fluents of these are $z =$

$t \times$ hyp. log. $\sqrt{\dfrac{r + x}{r - x}} + c$; which corrected by supposing $z = 0$

when $x = a$, are $z = t \times \left(\text{hyp. log. } \sqrt{\dfrac{r + x}{r - x}} - \text{hyp. log. } \sqrt{\dfrac{r + a}{r - a}}\right)$;

but $r \times \left(\text{hyp. log. } \sqrt{\dfrac{r + x}{r - x}} - \text{hyp. log. } \sqrt{\dfrac{r + a}{r - a}}\right)$ is the meri-

dional parts of the difference of the latitudes whose sines are x and

a, which call b; then is $z = \dfrac{tb}{r}$, the same as it is by Mercator's

sailing.

Further, putting $m = 2{\cdot}71828$ the number whose hyp. log. is 1,

and $n = \dfrac{2z}{t}$; then, when z begins at A, $m^n = \dfrac{r + x}{r - x}$, and there-

fore $x = r \times \dfrac{m^n - 1}{m^n + 1} = r - \dfrac{2r}{m^n + 1}$; hence it appears that as

m^n, or rather n or z increases (since m is constant), that x approxi-

mates to an equality with r, because $\dfrac{2r}{m^n + 1}$ decreases or converges

to 0, which is its limit; consequently r is the limit or ultimate value
of x: but when $x = r$, the ship will be at the pole; therefore the
pole must be the limit, or evanescent state of the rhumb or course:
so that the ship may be said to arrive at the pole after making an
infinite number of revolutions round it; for the above expression

$$\frac{2r}{m^n + 1}$$ vanishes when n, and consequently z, is infinite, in which case x is $= r$.

Now, from the equation $D = \dfrac{rd}{c} = \dfrac{sd}{r}$, it is found, that, when $d = 30° 57'$ the complement of the given latitude $59° 3'$, and $c =$ sine of $67° 30'$ the complement of the course, D will be $= 2010$ geographical miles, the required ultimate distance, which at the rate of 10 miles an hour, will be passed over in 201 hours or $8\frac{3}{8}$ days. The difference of longitude is shewn above to be infinite. When the ship has made one revolution, she will be but about a yard from the pole, considering her as a point.

When the ship has arrived infinitely near the pole, she will go round in the manner of a top, with an infinite velocity; which at once accounts for this paradox, *viz.* that though she make an infinite number of revolutions round the pole, yet her distance run will have an ultimate and definite value, as above determined: for it is evident that however great the number of revolutions of a top may be, the space passed over by its pivot or bottom point, while it continues on or nearly on the same point, must be infinitely small, or less than a certain assignable quantity.

IV. QUESTION 202, *by Mr.* Tho. Cooper.

There are two mountains, the higher situated in lat. 65°, and the other in lat. 64° 30'; the difference of their heights is 119 yards ; a straight line joining their summits is a tangent to the earth's surface and 63 miles in length. Required the time the sun first begins to illuminate the top of each mountain on the shortest day, reckoning that the earth's radius is 6980000 yards.

Answered by Mr. Rob. Heath.

Let $r =$ earth's radius $= co = cq = cn = 6980000$ yards; $d =$ diff. mountains heights $= 119$ yards, $x =$ height of the lowest mountain + earth's radius $= cm$? Then $\sqrt{(xx - rr)} = qm$, by 47 Eucl.I.

Again $\sqrt{(xx + 2dx + dd - rr)} = Mq$; but $Mq + qm = 63$ miles $= 110880$ yards $= b$. Consequently $\sqrt{(xx - rr)} - b = -\sqrt{(xx + 2xd + dd - rr)}$. Reduced, $xx + dx = \dfrac{bb - dd}{4} + \dfrac{bbrr}{bb - dd}$. In numbers $xx + 119x = 48723529707626\cdot7$, &c. Solved, $x = 6980164\cdot68$ yards; conse-

quently the height of the lowest mountain $=$ 164·68 yards; that of the highest $=$ 283·68 yards.
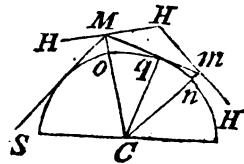
The time when the sun first begins to shine on the highest mountain (M) is when he cuts the tangent to the earth's surface (SM) below the horizon (HMH) in the angle HMS (found by trig.) $=$ 31ᵐ also he first begins to shine on the top of the lowest mountain (m) when he cuts the tang. (Mm) below the horizon (HmH) in the $\angle$ MmH $=$ 23ᵐ. Now if the sun's refraction be allowed for near the horizon $=$ 33', the angle HMS will be $=$ 1° 4'; and angle HmM $=$ 56ᵐ, which the sun is below the horizon of each mountain when he first shines on their tops.

The hour or times will be found by spheric trigonometry; thus, at 10ʰ 8ᵐ 38ˢ on the highest mountain, at 10ʰ 2ᵐ 4ˢ on the lowest.

N. B. Sun rises in lat. 65°, at 10ʰ 35ᵐ 16ˢ, in 64° 30', at 10ʰ 22ᵐ 56ˢ. This shews how much those are mistaken, who suppose the sun would first shine on them, nearly at his rising.

Additional Solutions.

In this problem we have given the base Mm, the perpendicular cq and the difference of the sides CM — cm, to construct the triangle which is reducible to one of the simpler cases of *Tangencies*, viz. to describe a circle through two given points to touch a circle given in magnitude and position. See Art. 4, Vol. I. of the Mathematical Repository.
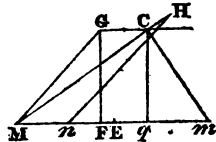
It may also be constructed as follows: Bisect the given base at E, and take Mn $=$ the given difference of the sides; then MC $+$ cm $=$ 2MC — Mn, and (2MC — Mn) Mn $=$ Mm.2Eq, Theor. 35, Cor. Vol. I. Course, or 2MC . Mn $=$ Mm.2Eq $+$ Mn². Make Mm . 2EF $=$ Mn²; then MC . Mn $=$ Mm . Fq, or MC : Fq :: Mm : Mn, a given ratio. Draw FG perpendicular and GC parallel to Mm, and draw nH parallel to MG meeting MC produced (if needful) in H: then GC $=$ Fq, and by similar triangles MC : GC $=$ Fq :: MH : Mn; therefore MH $=$ Mm. Whence this construction; having bisected the base Mm in E, take EF a third proportional to twice Mm, and Mn the difference of the sides; draw FG perpendicular to Mm and of a length $=$ to the given perpendicular; draw GC parallel to Mm and nH to MG; from M as a centre, with a radius equal to Mm, describe an arc to cut nH in H; draw MH cutting GC in c; then join mc and the triangle is constructed, the demonstration is obvious from the analysis.

The problem may also be resolved in the following manner.

If we conceive the triangle to be circumscribed by a circle, and a diameter drawn through the middle of the base, and a line through the vertex parallel to the base to intersect the diameter; then if v be put for the segment of the diameter above that line, and z for the

segment below the base ; p for the perpendicular, d for half the difference of the sides, and b for half the base ; we have, by well known theorems, $vz = d^2$ and $z(p + v) = b^2$; whence

$v : p + v :: d^2 : b^2$ or $v : p :: d^2 : b^2 - d^2$, a given ratio ; therefore v is given, consequently z is also given, and the method of construction is obvious.

The same Analytically.

Put $a =$ the perpendicular, $b =$ the base, $d =$ the difference of the sides, $\varphi =$ the angle at the vertex made by the greater side and the perpendicular, θ the angle made by the less side and the perpendicular ; then the sides will be denoted by a sec. φ and a sec. θ, and the segments of the base by a tan. φ and a tan. θ ; and we have

$$a \text{ tan. } \varphi + a \text{ tan. } \theta = b, \qquad a \text{ sec. } \varphi - a \text{ sec. } \theta = d,$$

Whence $a^2 \text{ tan.}^2\varphi = b^2 - 2ab \text{ tan. } \theta + a^2 \text{ tan.}^2\theta$

$$a^2 \text{ sec.}^2\varphi = d^2 + 2ad \text{ sec. } \theta + a^2 \text{ sec.}^2 \theta ;$$

subtracting the first equation from the second, and remarking that sec.^{2} — tan.^{2} = rad.^{2}, we have

$$a^2 = d^2 - b^2 + 2ab \text{ tan. } \theta + 2ad \text{ sec. } \theta + a^2,$$

or $2ad \text{ sec. } \theta = b^2 - d^2 - 2ab \text{ tan. } \theta$; and squaring

$$4a^2d^2 \text{ sec.}^2\theta = (b^2 - d^2)^2 - 4ab(b^2 - d^2) \text{ tan. } \theta + 4a^2b^2 \text{ tan.}^2\theta ;$$

or putting for sec.$^2 \theta$ its equal tan.$^2 \theta + 1$ and dividing by $4a^2(b^2 - d^2)$,

$$\text{tan.}^2\theta - \frac{b}{a} \text{ tan. } \theta = \frac{d^2}{b^2 - d^2} - \frac{b^2 - d^2}{4a^2},$$

therefore by completing the square, &c.

$$\text{tan. } \theta = \frac{b}{2a} \pm \frac{d}{2a} \sqrt{\frac{4a^2 + b^2 - d^2}{b^2 - d^2}}.$$

Whence the greater segment of the base $= \frac{b}{2} + \frac{d}{2}\sqrt{\frac{4a^2 + b^2 - d^2}{b^2 - d^2}},$

and the lesser segment $= \frac{b}{2} - \frac{d}{2}\sqrt{\frac{4a^2 + b^2 - d^2}{b^2 - d^2}}.$

<div align="right">J. L.</div>

v.   QUESTION 203, *by* J. B. S.

A ball of lead hanging from the top of a hall by a string, drawn over a pulley, which is 20 feet long between the centre of the ball and pulley, is set a swinging : The moment it begins to swing, a person, holding the other end of the string, begins to pull it, and draws up the ball, and continues so to do, at an uniform ratio of 5 feet in a minute, until he has pulled the ball quite up to the pulley. Quere, how many oscillations will the ball make before it reach the top ?

Answered by Mr. Ri. Dunthorne.

Let $a = 20$ feet, $b = 4$ minutes, $c =$ number of vibrations which the pendulum whose length is a makes in the time b, and $e =$ a small particle of time. Then $b : c :: e : \dfrac{ce}{b} =$ number of vibrations which the pendulum a makes in the time e; and $b : a :: e : \dfrac{ae}{b} =$ portion of the string drawn up in the time e; then will $a - \dfrac{ae}{b} =$ length of the pendulum after the first time e; $a - \dfrac{2ae}{b} =$ that after the second time e, &c. And $\dfrac{ccee}{b\,(b-e)} =$ the square of the number of vibrations in the second time e. Consequently its square root $=$ number of vibrations in the second time e. In like manner $\dfrac{ce}{\sqrt{b}\,\sqrt{(b-2e)}} =$ number of vibrations in the third time e. $\dfrac{ce}{\sqrt{b}\,\sqrt{(b-3e)}} =$ number in the fourth, &c. Whence it is manifest that the number of vibrations in the several times e, as above, are a series of fractions, whose numerators are equal, and their denominators are square roots, whose sides are single powers, decreasing in arithmetical progression from b, and $b \div e =$ the number of terms. So that by the arithmetic of infinites, $2c$ will be the sum of all the terms in the series. But $c = 97$, whence $2c = 194$, the number of vibrations required.

See also Question 681.

VI. QUESTION 204, by Mr. Mason.

Soon after a hard gale of wind, some persons strolling along shore, upon the look-out, found a large cask just driving ashore, which proved a piece of old Jamaica rum :. They soon boarded it, and racked off forty-one gallons, and filled up the cask again with water, and acquainted some more of their party with their success, who went and racked off the same quantity of the mixture, and filled it up again with water: In like manner it was served so twice more; and at last, by the proof, there was found $25\frac{544}{1255}$ gallons of rum remaining, the rest water. How much did the whole piece contain, and how many gallons of rum was drawn out at each evacuation?

Answered by Mr. Rob. Heath.

Let $b = 25\frac{914}{1000} = 25{\cdot}2935528$ gallons of rum left in the cask.
  $q = 41$ gallons, the liquor drawn off each time.
  $n =$ the number of times of drawing.   And
  $x =$ the quantity of neat liquor the cask held.

Say $x : x - q :: x - q : \dfrac{(x - q)^2}{x} =$ rum left at 2d drawing.

$x : \dfrac{(x - q)^2}{x} :: x - q : \dfrac{(x - q)^3}{xx} =$ rum left at 3d drawing.

$x : \dfrac{(x - q)^3}{xx} :: x - q : \dfrac{(x - q)^4}{xxx} =$ rum left at 4th drawing.

Whence it is evident, that the quantity of neat liquor, left at any number of times drawing off, will be universally $\dfrac{(x - q)^n}{x^{n-1}}$; conse-

quently $\dfrac{(x - q)^4}{xxx} = b$.   Reduced $x - q = b^{\frac{2}{4}} x^{\frac{3}{4}}$.   In   numbers

$x - 41 = 2{\cdot}1426x^{{\cdot}75}$; solved, [according to a new method of managing exponentials] $x = 124{\cdot}671$, &c.

Now, if each quantity left in the cask, at any time of drawing off, be subtracted from the quantity left the time preceding, the neat rum at each time drawn off will be found ; and is as follows:

Rum drawn off 1.   41·000 ⎫
               2.   27·516 ⎪
               3.   18·467 ⎬ gallons
               4.   12·394 ⎪ Total 124·67.
  add gallons left ....   25·293 ⎭

THE PRIZE QUESTION, *by Mr.* Ri. Dunthorne.

Let AFK be the conchoid of Nicomedes, which is continually approaching nearer to the line BC, yet if continued *ad infinitum* could never meet; which at first setting out to generate the line AFK, the distance BA is 16 inches ; and the distance BP is 24 inches from the pole P to the asymptote BC.   It is required to find the distance of F, the point of inflection of the curve, from the line BC?
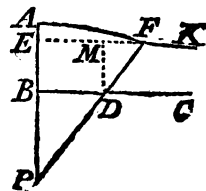
Answered by Mr. Hen. Travis.

Let PB and AB be called b and c, the abcissa and ordinate x and

y; then $bbcc + 2bccy + ccyy - bbyy - 2byyy - y^4 = yyxx$, will be the equation, expressing the nature of this curve, &c. The first in fluxions, $2bcc\dot{y} + 2cc y\dot{y} - 2bb y\dot{y} - 6byy\dot{y} - 4y^3\dot{y} = 2yy\dot{x}xx + 2xx\dot{y}yy$; again by an uniform velocity $ccy\dot{y} - 6byy\dot{y} - 6yyy\dot{y} = yy\dot{x}xx + 2yy\dot{x}xx + 2yy\dot{x}xx + \dot{x}xyy$; and from the 1, 2, and 3 steps we have $y^3 + 3byy = 2bcc$. Therefore $y = 12 \cdot 0885$ the answer.

Answered by Mr. J. May.

Put AB or DF $= 16 = a$, BP $= 24 = b$, BE or DM $= x$, and EF $= y$. Then by the similar triangles DMF and PEF, we have DM $= x$: MF $= \sqrt{(aa - xx)}$ :: PE $= b + x$: EF $= y$. Multiplying extremes and means, and dividing by x, we have $y = \dfrac{(b + x) \sqrt{(aa - xx)}}{x}$; in

fluxions $\dot{y} = \dfrac{(x^3 + aab)\dot{x}}{xx\sqrt{(aa - xx)}}$; again $\ddot{y} = \dfrac{(2a^4b - aax^3 - 3aabxx) \times \dot{x}x}{(aax^3 - x^5)\sqrt{(aa - xx)}}$ $= 0$. Then is $x^3 + 3bxx - 2aab = 0$; by means of this equation the value of x may be found by the section of a parabola and circle. But if brought into numbers, and reduced, will give $x = 12 \cdot 0885$ inches for BE; being the distance of the point of inflection of the curve from the asymptote.

Questions proposed in 1739, and answered in 1740.

I. QUESTION 205, by Mr. John Turner.

There is a piece of ground in the form of a triangle, as ACB, side AC $= 14$, and side BC $= 22$ chains; and a circular drain CE, centre A, radius AC, cuts the side AB in E so that 2 AB $= 5$CE. Required AB ?

Answered by the proposer, Mr. J. Turner.

Let AB $= x$, AC $= 14 = b$, CB $= 22 = c$, $cc - bb = 288 = m$. Then the cosine of the angle BAC $= (xx - m) \div 2x$: but the arch CE $= \cdot 4x = nx$; and the radius $= b$; consequently the cosine of the said arch $= b - \dfrac{nnxx}{2b} + \dfrac{n^4x^4}{24b^3} - \dfrac{n^6x^6}{720b^5}$, &c. Therefore $\dfrac{xx - m}{2x} = b - pxx + qx^4$, &c. From which equation by converging series, the value of

s 3
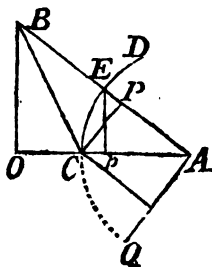
x comes out $= 28\cdot9443$: And the angle BAC $= 47$ degrees 17 min.

Merones *answers this question thus:*

Let the radius AC $= r$, CB $= s$; take an arch p, as near CB as possible, let $a =$ its sine, $b =$ its cosine to the radius r; and let $p + z =$ CB : and per quest. $\frac{5}{2}p + \frac{5}{2}z =$ AB; also the cosine AP $= b - \frac{az}{r} - \frac{bz^2}{2rr} + \frac{az^3}{6r^3} +$ &c. And BA $-$ AP $= \frac{5}{2}p + \frac{5}{2}z - 2b +$ $\frac{2a}{r}z + \frac{b}{rr}z^2 - \frac{az^3}{3r^3} -$ &c. different segments of the base. But by Ax. 4th of plain trigonometry, $\frac{5}{2}p + \frac{5}{2}z : s + r :: s - r :$ different segments. Whence we have

$$\begin{array}{ll} +\dfrac{5ap}{r} & +\dfrac{5bp}{2rr} \\ +\dfrac{25p}{2}z + \dfrac{5a}{r}z^2 & -\dfrac{5ap}{6r^3} \\ -5b & +\dfrac{25}{4} \end{array} +\dfrac{5b}{2rr}\,z^3 = \begin{array}{l} + ss \\ - rr \\ + 5bp \\ - \frac{25}{4}pp \end{array} = \text{R.}$$

Now assume $p = 11\frac{1}{2}$: and then $\dfrac{180p}{r \times 3\cdot141592} =$ the degrees of the arch P; from whence is had $a = 10\cdot249675$; $b = 9\cdot536464$; all which substituted in the foregoing series, and putting A, B, C, &c. for the known co-efficients, and reversing the series, we shall have $z = \dfrac{R}{A} - \dfrac{BR^2}{A^3} + \dfrac{2BC - AC}{A^5}R^3$, &c. $= 0\cdot704043$, and he arch CE $= 11\cdot5704043$; and AB $= 28\cdot926011$.

Mr. Robert Heath

Says, by a table of natural sines and a few trials, I find the angle BAB $= 47^\circ 21'$; whence the required side AB $= 28\cdot925$, &c. The method of solving this by infinite series, which converges so slow, renders it more tedious than useful.

Answered by Mr. Hen. Travis.

Let CB $= b$, CA $= a$, AP $= y$; and as in Simpson's Flux. p. 121, we have the arch QC $= y + \dfrac{y^3}{2.3a^2} + \dfrac{3y^5}{2.4.5a^4} + \dfrac{3\cdot5y^7}{2.4.6.7a^6}$, &c. and $\dfrac{2a \times 3\cdot1416}{4} =$ the side QB $= 21\cdot991 = q$; from which take QC,

leaves $q - y - \dfrac{y^3}{2.3a^2} - \dfrac{3y^5}{2.4.5a^4} - \dfrac{3.5y^7}{2.4.6.7a^6}$, &c. $=$ cE; which

multiplied by $\dfrac{5}{2}$ gives $\dfrac{5q}{2} - \dfrac{5y}{2} - \dfrac{5y^3}{2.2.3a^2} - \dfrac{3y^5}{2.4a^4}$, &c. $=$ AB, and

per Eucl. 1. 47, $aa - yy =$ cF^2; also $bb - aa + yy =$ BF^2, or

$288 + yy =$ BF^2: Call $288 = rr$; then FB $= \sqrt{(rr + yy)}$, or $r +$

$\dfrac{yy}{2r} - \dfrac{y^4}{8r^3} + \dfrac{y^6}{16r^5}$, &c. $=$ FB; to which add y, gives $r + y +$

$\dfrac{y^2}{2r} - \dfrac{y^4}{8r^3} + \dfrac{y^6}{16r^5}$, &c. $=$ AB: hence $r + y + \dfrac{y^2}{2r} - \dfrac{y^4}{8r^3} +$

$\dfrac{y^6}{16r^5}$, &c. $= \dfrac{5q}{2} - \dfrac{5y}{2} - \dfrac{5y^3}{2.2.3a^2} - \dfrac{3y^5}{2.4a^4}$; therefore $y + \dfrac{5y}{2} +$

$\dfrac{y^2}{2r} + \dfrac{5y^3}{2.2.3a^2} - \dfrac{y^4}{8r^3} + \dfrac{3y^5}{2.4a^4}$, &c. $= \dfrac{5q}{2} - r$; multiplying by 2,

and dividing by 7, gives $\dfrac{5q - 2r}{7} = z$; $\because\ y = z - \dfrac{zz}{7r} + \dfrac{2z^3}{49r^3} -$

$\dfrac{5z^3}{7 \times 2a^3} + \dfrac{5 \times 5z^4}{7 \times 7 \times 2ra^3} - \dfrac{5z^3}{7 \times 7 \times 7r^3} + \dfrac{z^4}{7 \times 4r^3}$, &c. $\because\ y =$

$9\cdot46$ and AB $= 28\cdot94$.

Mr. Nich. Farrer's *answer.*

Let fall the perpendicular BO, on AC produced, also Ep from the point E; then let $r =$ AE $= 14$, $m =$ BC $= 22$; and put $x =$ the arch of a circle, whose radius is unity, similar to the arch EC; then

$rx =$ arch EC per similar figures; and its sine $= r \times x - \dfrac{x^3}{6} +$

$\dfrac{x^5}{120} - \dfrac{x^7}{5040}$, &c. $=$ Ep; and $\dfrac{5rx}{2} =$ AB per question, and $\dfrac{25rxx}{8}$

$- \dfrac{mm + rr}{2r} =$ oc per 12 Eucl. 2. substitute $a = \dfrac{25r}{8}$, and $b =$

$\dfrac{mm + rr}{2r}$; then $axx - b =$ oc; and per 47 Eucl. 1,

$\sqrt{(mm - aax^4 + 2baxx - bb)} =$ BO; and per 4 Eucl. 6, AE : Ep ::

AB : BO, i. e. $r : r \times x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$, &c. :: $\dfrac{5rx}{2} : \sqrt{(mm - aax^4 +}$

$2baxx - bb)$; therefore $\dfrac{2}{5rx} \sqrt{(mm - aax^4 + 2baxx - bb)} = x -$

$\dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040}$, &c. and the value of $x = \cdot82646$; and AB $=$

$28\cdot9211$ chains required in answer.

II.   QUESTION 206, *by* Merones.

If a cannon ball be projected upwards in a direction perpendicular to the horizon, half a mile high, and in the latitude of 53 degrees; where will it fall?

<center>*Answered by* Merones.</center>

Let the ball be projected from A, the time of its flight will be $25\frac{3}{4}$ seconds, in which time the point A will be carried to B, through a space AB, of 23666 feet, by the earth's rotation. Now the ball (carried by a compound motion of its projection and the earth's rotation) will describe an ellipsis whose focus is in the centre of the earth; in which the elliptic area A*rt*c = the circular sector ABC: Or the area A*rt*A = area BC*t*; but by reason of the small ratio of *rs* to AC the portion A*rt* may be taken for a parabola. Let AC = b = 21000000 feet, AB = d = 23666 feet, *rs* = h = 2640, a = B*t*; then will $\frac{1}{2}ba = \frac{2}{3}h$ $(d-a)$; and $3ba + 4ha = 4hd$; whence $a = \dfrac{4hd}{3b+4h} = \dfrac{4hd}{3b}$ nearly = 3·967 feet. Near 4 feet to the west.
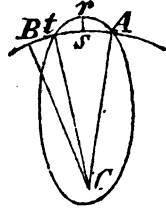
<center>*Mr.* Hen. Travis's *answer*.</center>

The time of the ball's ascending is equal to the time of its descending, according to the writers on projectiles; which time call (x) and the number of feet a heavy body will fall or descend freely, by the force of its own gravity, in one second of time = n. Then will nxx = 2640 = the feet in half a mile; ∵ $x^2 = \dfrac{2640}{n} = \dfrac{2640}{16\cdot1}$ = 163·98 nearly, ∵ x = 12′ 48″ = the time the ball is ascending or descending; and consequently the ball will fall near 4 feet from the place it was projected.

<center>III.   QUESTION 207, *by Mr.* John May, *jun.*</center>

There came three Dutchmen of my acquaintance to see me, being lately married; they brought their wives with them. The men's names were Hendrick, Claas, and Cornelius; the women's, Geertruii, Catriin, and Anna: but I forgot the name of each man's wife. They told me they had been at market to buy hogs; each person bought as many hogs as they gave shillings for each hog: Hendrick bought 23 hogs more than Catriia, and Claas bought 11 more than

Geertruii ; likewise, each man laid out 3 guineas more than his wife. I desire to know the name of each man's wife ?

Answered by Mr. J. Hill.

Call the number of hogs any woman bought x; the number her husband bought $x + n$; money laid out by the woman is xx shillings ; money laid out by the husband is $xx + 2nx + nn$ shillings. Equation $xx + 2nx + nn = xx + 63.$ $\therefore x = \dfrac{63 - n}{2n}.$ If $n = 1$, then $x = 31$, and $x + n = 32$: hence some woman bought 31 hogs, and her husband 32 : If $n = 3$, then $x = 9$, and $x + n = 12$; therefore some other woman bought 9, and her husband 12: If $n = 7$, then $n + x = 8$; $\therefore$ some woman bought 1, and her husband 8.   Consequently

$$\begin{aligned}
&\text{Hendrick bought } 32, \text{ and his wife Anna} \quad 31 \\
&\text{Claas} \ldots\ldots\ldots\ldots \ 12, \ldots\ldots\ldots \text{ Catriin} \quad 9 \\
&\text{Cornelius} \ldots\ldots \ 8, \ldots\ldots \text{Geertruii} \quad 1
\end{aligned}$$

Answered by Merones.

	Men.			Women.		
For the persons put	A,	B,	C,	P,	Q,	R,
Hogs	$a,$	$e,$	$y,$	$e - c,$	$a - b,$	$u,$
Money	$aa,$	$ee,$	$yy,$	$(e - c)^2,$	$(a - b)^2,$	$uu.$

Let $b = 23$, $c = 11$. Compare B with Q, then per question $ee - (a - b)^2 = 63$ shillings ; that is, putting $e = a + z$; $2az + zz + 46a = 592$; therefore $a = \dfrac{23 - z}{2} + \dfrac{63}{2z + 46}$; now it is evident the last term cannot be a whole number ; therefore z in the first term must be an even number ; so the last term $\dfrac{63}{2z + 46}$ must be the half of a whole number ; let $\dfrac{63}{z + 23} = v$.   Whence $z = \dfrac{63}{v} - 23$; hence v must be either 1, 3, 7, 9, 21, or 63; from each of which is had $\begin{cases} a \ 54, \ 32, \ 14, \ 22, \ 24 \\ e \ 32, \ 12, \ 12, \ 8, \ 8 \end{cases}$. And again comparing c with P, then $yy - ee + 22e = 184$; and we find $\begin{cases} y \ 12, \ 8, \ 8, 42, \ 32. \\ e \ 2, \ 10, \ 12, \ 20, \ 42. \end{cases}$

Whence e must be the same in both suppositions ; therefore it is 12, if the question be possible in whole numbers.   But since the other two persons A, R, must be compared, therefore $aa - uu = 63$: From hence $a = 32$, $u = 31$, $e = 12$, and $y = 8$; but comparing the men and women in any other manner, it will appear there is no other answer in whole numbers.   Therefore Hendrick and Anna, Claas and Catriin, and Cornelius and Geertruii, are man and wife.

The same answered by Mr. Rob. Heath.

Let $x =$ the hogs bought by either Hendrick, Claas, or Cornelius ; then xx will be the shillings they cost, and $xx - 63$ the shillings their wives hogs cost, which (as whole hogs) must always be a square number ; because the square root of the shillings laid out for each parcel is equal to the number of hogs. Let $x - y =$ the side of that square, then $xx - 63 = xx - 2xy + yy$. Consequently, by reduction, $x = \dfrac{63 + yy}{2y}$; whence we find y may be

Hogs.

$$
\left.\begin{array}{c}1\\3\\7\end{array}\right\} \text{Conseq.} \quad x = \left\{\begin{array}{c}32\\12\\8\end{array}\right\} \begin{array}{l}\text{bought by the}\\\text{men, coupled}\\\text{with}\end{array} \left\{\begin{array}{c}31\\9\\1\end{array}\right\} \text{bought by their wives.}
$$

Whence are joined Hendrick and Anna, Claas and Catriin, Cornelius and Geertruii.

Mr. N. Farrer

Observes, that the number of hogs the three men and their respective wives bought will be expressed by three pair of numbers, the difference of whose squares must be 63. Now all the whole numbers whose squares will produce this difference are 1 and 8, 9 and 12, 31 and 32 ; there 8, 12, 32, the men bought ; 1, 9, 31 the women.

IV. QUESTION 208, by Mr. Hen. Travis.

Given this equation, viz. $Axy + Bu + z - c = 0$; expressing the relation of the sides of a trapezium inscribed in a circle whose diameter is known to be 75 feet (or d). Required the sides separately, and area, by a general method that will resolve all such problems ?

N. B. $A = 100$; $B = 5$; $c = 432246$.

Answered by the Proposer.

$$d : y :: z : \frac{zy}{d} = CE. \quad \text{Hence}$$

$$\sqrt{\left(yy - \frac{zzyy}{dd}\right)}, \quad \text{or} \quad \frac{\sqrt{(ddyy - zzyy)}}{d} = BE,$$

$$\sqrt{\left(zz - \frac{zzyy}{dd}\right)}, \quad \text{or} \quad \frac{\sqrt{(ddzz - zzyy)}}{d} = ED,$$

$$\frac{\sqrt{(ddzz - zzyy)} + \sqrt{(ddyy - zzyy)}}{d} = BD.$$

$\sqrt{(ddxx - uuxx)} + \sqrt{(dduu - xxuu)} =$ BD,

Axy + Bu + z = c, per quest.

z = c — Axy — Bu; which substit. for z.

$\sqrt{(ddxx - uuxx)} + \sqrt{(dduu - xxuu)} =$

(c — Axy — Bu) $\sqrt{(dd - yy)} +$

$\sqrt{(ddyy - yy}$ (c — Axy — Bu)^2).
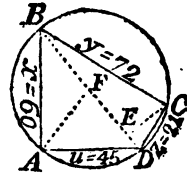
Here we have one equation including three unknown quantities, and yet the question is truly limited; and to be resolved as the following question is:
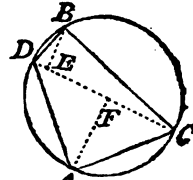
Given $2yx + \dfrac{2592}{uy} + 4ux = 48x - x^2$.

Quere x, y, and u? In fluxions,

$2y\dot{x} + 4u\dot{x} = 48\dot{x} - 2x\dot{x}$,

$2\dot{y}x - \dfrac{2592u\dot{y}}{yyuu} = 0$, and $- \dfrac{2592\dot{u}y}{uuyy} + 4\dot{u}x = 0$.
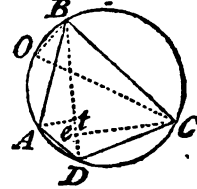
From the first, second, and third steps, by common algebra we get $x = 12$, $y = 6$, and $u = 3$; and by the very same method of reasoning, the sides of the trapezium are found to be $x = 60$, $y = 72$, $u = 45$, and $z = 21$; and the area = 2106 square feet. This trapezium may be placed 4 or 5 different ways in a circle, which I have proved by a large geometrical projection, and every way justly contain the remaining chords of the circle, and measured, amount to each the same area 2106 square feet. (1) In the first fig. the several lines are obvious; and the diagonal BD = 75, multiplied by the half of the two perpendiculars (CE = 19·65, AF = 36·52) will give the area as above. (2) In the second fig. BC = y = 72, BD = z = 21, DA = x = 60, and AC = u = 45. And the diagonal DC (=74·96) multiplied by half the perpendiculars (AF = 35·75, and BE = 20·2) gives nearly the same area. (3) In the third fig. y is the line CB, x is AB, z is AD, and u is x. The diagonal BD = 70·01, the perpendicular At = 16·6, ce = 43·1 nearly; and (4) in the same fig. z is represented by BO, u by OA, and x by AC; the diagonal OC is nearly 74·95, which by the perpendiculars gives the area as before.

Mr. Paul Sharp has found the sides 72, 60, 45, and 21, in answer.

Mr. Tho. Robinson gives the sides 72, 2, 59, 9, 45, 1, 20, 7, nearly true. In this question it does seem to appear, that the number of quantities sought exceed the number of given equations, and (as my ingenious correspondents have observed) is unlimited. But I presume since the numbers given in the question, viz. A = 100, B = 5,

and $c = 432246$; and the four numbers sought are together ob-liged to extend the chords of 360 degrees, and the diameter of the circle is given; it may be said to be limited; but I shall rather leave it to the speculation of those ingenious persons who are pleased to appear in the emendata next year.

v. QUESTION 209, *by Mr*. Robert Heath.

Find x^2, when $\left(x^{\frac{1}{3}}\right)^{x^{\frac{2}{3}}}$ is a minimum.

Answered by Merones.

$\left(x^{\frac{1}{3}}\right)^{x^{\frac{2}{3}}}$ = a minimum. Therefore $x^{\frac{2}{3}} \times$ log. $x^{\frac{1}{3}}$ minimum; whence $\frac{2}{3}x^{-\frac{1}{3}}\dot{x} \times$ log. $x^{\frac{1}{3}} + \frac{1}{3}x^{-\frac{1}{3}}\dot{x} = 0$; or 2 log. $x^{\frac{1}{3}} + 1 = 0$. And therefore hyp. log. $x^{\frac{1}{3}} = -\frac{1}{2}$. And tab. log. of $x^{\frac{1}{3}} = -\frac{1}{2} \times$ ·434294 $= -$ 1·782853. Therefore $x^{\frac{1}{3}} = $ ·60653, and $x = $ ·22313.

The Proposer, Mr. Heath, answers thus.

Its log. is a minimum. Let $y^3 = x$, the expression will become y^{y^2}; its log. $= $ l. $y \times yy$, fluxed, $\dot{y}y + $ l. $y \times 2y\dot{y} = 0$. Reduced, l. $y = -\frac{1}{2}$, consequently l. $y^3 = $ l. $x = -\frac{3}{2}$: Whence the natural number corresponding thereto is $= $ ·223130, &c. and accurately, which is the value of x required.

VI. QUESTION 210, *by Mr*. Rich. Dunthorne.

Suppose a cask in form of the middle frustum of an hyperbolic spindle, whose length is 24 inches, bung diameter 30, head diameter 20, and transverse axis of the generating hyperbola 100 inches. Required its content in ale gallons?

Answered by Mr. Rob. Heath.

In the hyperbola, there is given the transverse axis in the figure $= $ 100 inches $= p$, BG $= 15 = a$, HD $= m$G $= 10$, B$m = $ $5 = b$, GD $= m$H $= 12 = c$; let $x = $ BO, and $y = $ OH required. By the property of the curve, $(p + b) b : cc :: (p + x) x : yy$; $\therefore xx + px$ $= \dfrac{p + b}{cc} byy$, and $x = \sqrt{(\tfrac{1}{4}pp + \dfrac{p + b}{cc}byy)}$
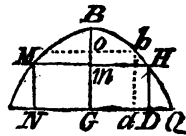
$-\frac{1}{2}p$; $a - x = a + \frac{1}{2}p - \sqrt{(\frac{1}{4}pp + \dfrac{p+b}{cc}\,byy)} = $ oG $=$ hd,

(put $f = a + \frac{1}{2}p$, $g = \dfrac{p+b}{cc}\,b$); it is plain the fluxion of Gd $\times$ into the area of the circle whose radius is hd, is equal the fluxion of the indefinite solid generated by the rotation of the curve (Bh) about Gd: $ff + \frac{1}{4}pp + gyy - 2f\sqrt{(\frac{1}{4}pp + gyy)}$: $\times$ $3\cdot1416\dot{y}$; whose fluent is $3\cdot1416y$ $(ff + \frac{1}{4}pp) + 3\cdot1416y^3 \times \frac{g}{3}$ (let $n = 3\cdot1416$) —

$fpny - \dfrac{2fng\,y^3}{3p} + \dfrac{2fng^2y^5}{5p^3} - \dfrac{4fng^3y^7}{7p^5} + \dfrac{10fng^4y^9}{9p^7}$ — &c. and if c be put instead of y, in the expression, we have the solidity of half the cask $= 6754\cdot8$ inches, true to a decimal; consequently $13509\cdot6$ inches the whole content, or $47\cdot893$ ale gallons.

<center>Merones's answer.</center>

Let the semi-transverse CA $= r = 50$, CD $= a = 65$, the rect-angle QPA $= pp = 525$, PE $= q = 12$; $dd = aa + rr$, $c = 3\cdot14159$; CB $= x$, GF or DB $= a-x$, BF or GD $= y$. Then per conics, $qq : pp :: yy : xx - rr$; whence $x = \sqrt{(rr + (pp \div qq)\,yy)}$; and BD$^2 = (a-x)^2 = dd + (pp \div qq)\,yy - 2a\sqrt{(rr + (pp \div qq)yy)}$: Therefore DB$^2 \times c\dot{y} = cdd\dot{y} + c(pp \div qq)\,yy\dot{y} - 2cay\sqrt{(rr + (pp \div qq)\,yy)}$ $=$ flux. solid AFGD revolving round GD; put L $= 2\cdot302585 \times$ log. $\dfrac{p + \sqrt{(rr + pp)}}{r}$; and write q for y in the fluent, and we shall have

$cddq + \frac{1}{3}\,cppq - caq\sqrt{(rr + pp)} - \dfrac{caqrr}{p}$ L $=$ half the cask $=$ $6754\cdot88$ inches, and the whole $= 13509\frac{3}{4} = 47\cdot907$ ale gallons.
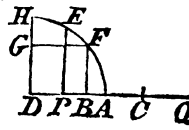
<center>VII.   QUESTION 211, by Mr. Ri. Lovatt.</center>

Suppose that in the spherical triangle ABD, there is given AB $= 80^\circ$ $3'$, AD $= 60^\circ$ $10'$, AC $= 40^\circ$ $21'$, the angle BAD $= 73^\circ$ $0'$, and DCA $=$ BCA. What are the sides DC and BC.
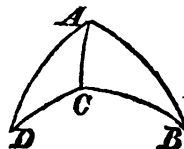
<center>Answered, by Mr. R. Heath.</center>

Given	°	′	Sines	Cosines	
AD	= 60	10	·86747 = b	·49747 = c	∠DCA =
AC	= 40	21	·64745 = d	·76210 = f	ACB
AB	= 80	3	·98495 = g	·17278 = h	Required DC
∠DAB	= 73	0	·95630 = s	·29237 = t	and CB?

It must be noted that DC and CD are not two continued arches, for so the question would be over-limited and absurd, but sides of different triangles. Let $x = \cos \angle$ DAC, $\sqrt{(1 - xx)}$ its sine, radius $= 1$. Then (per Auderson's Theor.) $bdx + cf = \cos.$ DC; its sine $\sqrt{(1 - (bdx + cf)^2)}$. By trigonometry, s. DC : s. $\angle$ DAC :: s. DA : s.
DCA $= b \sqrt{\dfrac{1 - xx}{1 - (bdx + cf)^2}}$: Now cos. $\angle$ CAB $= xt + s \sqrt{(1 - xx)}$,
its sine $xs - t \sqrt{(1 - xx)}$ and by the aforesaid theorem $dgtx + dgs$ $\sqrt{(1-xx)} + hf =$ cosine CB. Whence $\sqrt{(1 - (dgtx + dgs \sqrt{(1-xx)} + hf)^2)}$ $=$ its sine : Per trigonometry, s. CB : s. $\angle$ CAB :: s. AB : s. $\angle$ ACB
$= \dfrac{xs - t\sqrt{(1 - xx)}}{\sqrt{(1 - (dgtx + dgs\sqrt{(1-xx)} + hf)^2)}} \times g = b \sqrt{\dfrac{1 - xx}{1 - (bdx + cf)^2}}$.
By reduction, $(1 - (bdx - cf)^2) \times (xs - t \sqrt{(1 - xx)})^2 \times g^2 =$ $bb \times (1 - xx) \times (1 - (dgtx + dgs \sqrt{(1 - xx)} + hf)^2)$. In numbers, $(1 - (\cdot561643x + \cdot379121)^2) \times (\cdot941907x - \cdot287969 \sqrt{(1-xx)})^2 =$ $\cdot7525 \times (1-xx) \times (1 - (1847x + \cdot60993\sqrt{(1 - xx)} : + \cdot131675)^2)$. Here x [by a new method of solving equations] is found $= \cdot8770$, &c. Whence DC $= \cdot4900 = 29° 21'$, and BC $= \cdot8086 = 53° 58'$.

Mr. *J. Turner*, observes the scheme is false drawn, and so makes no question at all; but correcting it, and putting $b =$ rectangle s. AD, AC; $c =$ their cosines; $d =$ rectangle AC, AB; $f = \cos. g =$ s. AD; $h =$ s. AB; $m =$ s. $\angle$ BAD, $n =$ its cos. $x =$ cos. $\angle$ DAe, $\sqrt{(1-xx)} =$ its sine. Then the sine of BAC $= mx - n \sqrt{(1 - xx)}$, and its cos. $= nx + m$ $\sqrt{(1 - xx)}$; cos. DC $= bx + c$; cos. BC $= dnx + dm \sqrt{(1-xx)} + f$. Sine DC $= \sqrt{(1 - cc - 2bx - bbxx)}$; s. BC $=$
$\sqrt{(1 - ddnnxx - 2dnfx - ff - ddmm + ddmmxx - 2ddnmx - 2dmf\sqrt{(1-xx)})}$.
As $1 - cc - 2cbx - bbxx : 1 - xx :: gg : \dfrac{gg - ggxx}{1 - cc - 2bcx - bbxx}$
$=$ square s. $\angle$ DCA. Again, BC^2 : s.^{2} $\angle$ BAC :: BA^2 : s.^{2} $\angle$ DCA.
Consequently $\dfrac{gg - ggxx}{1 - cc - 2bcx - bbxx} =$
$\dfrac{hhmmxx - 2hhmnx \sqrt{(1 - xx)} + hhnn - hnx}{1 - ddnnxx - 2dnfx - ff - ddmm + ddmmxx - 2ddnmx \sqrt{(1-xx)} - 2dmf\sqrt{(1 - xx)}}$;
which when all the terms affected with $\sqrt{(1 - xx)}$ are brought to one side of the equation, and involved, will produce an equation, of the 8th power; in which $x = \cdot87719$. Consequently the $\angle$ DAC $=$ $28° 42'$; BAC $= 44° 18'$; side DC $= 29° 19'$; and BC $= 53° 59'$.

VIII. QUESTION 212, *by Mr.* Chr. Mason.

There is a triangular piece of ground, whose centre of gravity measures from each angle 12, 16, and 20 chains : It is required to find the periphery of the greatest inscribed ellipsis ; and also the content of each angular piece without the ellipsis ?

Answered.

There is no solution printed to this question this year; but in the Emendations in the next year the Diary Author says, the printer omitted it for want of room, and that of several who answered it, Mr. Hill's numbers for the sides of the triangle are 34·151, 28·521, and 19·549.

Again in the Emendations in the year 1744 he mentions it, saying that Mr. Heath had fully answered it at first; and that Mr. James Terey now puts the side sought $= x$, from one angle to the intersection $= d$, another $= c$, the 3d $= b$; then $x = \sqrt{(2dd + 2cc - bb)}$: Whence he gives the sides 34·176, 28·844, and 20; the area $= 288$. Then $1 : ·6046 ::$ area of the equilateral $\triangle$: area inscr. circle. Hence area of inscr. ellipsis $= 174·124$, and each angular piece $= 37·95$; diam. inscr. circle 10·88 $=$ conjugate of the ellipse; longest 20·36.

Solution.

The method of finding the sides of the triangle may be thus: Put x, y, and z for the three sides, and a, b, c, ($= 12$, 16, and 20) for the distances between the angular points and the centre of gravity; then because the lines a, b, c, if produced would bisect the opposite sides, and are each $\frac{2}{3}$ of the whole bisecting line, by a known theorem in geometry, we have

$$x^2 + y^2 - \tfrac{1}{2}z^2 = \tfrac{9}{2}c^2, \; x^2 + z^2 - \tfrac{1}{2}y^2 = \tfrac{9}{2}b^2, \; y^2 + z^2 - \tfrac{1}{2}x^2 = \tfrac{9}{2}a^2\,;$$

the sum of these being taken, and cleared of fractions, we have $xx + yy + zz = 3aa + 3bb + 3cc$ [which is a very curious theorem]; from this last each of the three former equations being subtracted, &c. we have

$$x = \sqrt{(2bb + 2cc - aa)} = 4\sqrt{73} = 34·176,$$
$$y = \sqrt{(2aa + 2cc - bb)} = 4\sqrt{52} = 28·8444,$$
$$z = \sqrt{(2aa + 2bb - cc)} = 4\sqrt{25} = 20.$$

Otherwise. Or the triangle might be easily constructed. For, since the three lines drawn to the centre of gravity divide the whole triangle into three equal parts or triangles, and each triangle being equal to half the sine of its angle at the centre of gravity drawn into the product of the two sides or lines about it, therefore the sines of the three angles formed by those three given lines are reciprocally as the products of each two about them; but those sides being 12, 16, and 20, are as 3, 4, and 5; therefore the sines of the angles are as

$$\frac{1}{3 \times 4}, \quad \frac{1}{3 \times 5}, \quad \text{and} \quad \frac{1}{4 \times 5};$$

or as 5, 4, and 3; and so universally the sines of the three angles are always as the three given lines. But the three angles about one point are equal to four right angles, or 360°; whence the problem is to divide 360°, or a circle into three such parts, that their sines shall obtain the given ratio. In the present case the ratios 5, 4, and 3 form a right-angled triangle; let

it be ABC; then the supplements of the $\angle$s A, B, and C, of this triangle are the angles required to be formed by the given lines at the centre of gravity. For, with the centre A and radius AC describe the circle; then it is evident that CB is the sine of the arc DC or $\angle$DAC, that AB or CE is the sine of CF or $\angle$CAF, and that AC is the sine of the remaining quadrant FD or $\angle$FAD; and these are respectively the supplements of the three angles of the triangle ABC.
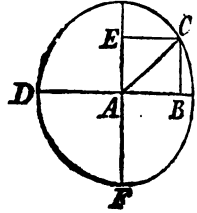
The other parts of this question will be done as quest. 430, proposed in the year 1757, where the subject is resumed, and to which therefore we refer. H.

THE PRIZE QUESTION, *by Mr.* R. Heath.

Given the latitude of three places, Moscow 55° 30′, Vienna 48° 12′, Gibraltar 35° 30′, all lying directly in the same arc of a great circle: The difference of longitude between Vienna (situated in the middle) and Moscow, easterly, is equal to that between Vienna and Gibraltar, westerly: It is required to find the true bearing and distance of each place from the other, and the difference of longitude, according to the convexity of the globe?

Answered by Mr. J. Turner.

Let P represent the pole of the world; M, Moscow; V, Vienna; G, Gibraltar. Put $x = \cos. \angle$GPV $= \angle$VPM; $\sqrt{(1 - xx)} =$ its sine; sine GP $= b$; of PM $= c$; s. $\angle$GVP $= \sqrt{(1 - xx)} \div z$; the rectangle of the sines of GP, PV $= d$; rect. of cos. $= f$; rectangle of the sines of VP, PM $= g$; rect. cosines $= h$. By Auderson's Theorem, $dx + f = \cos.$ GV, and $gx + h = \cos.$ VM; as $\sqrt{(1 - xx)} \div z : b :: \sqrt{(1 - xx)} : bz =$ sine GV, and its cos. $= \sqrt{(1 - bbzz)}$; as $\sqrt{(1 - xx)} \div z : c :: \sqrt{(1 - xx)} : cz =$ sine VM, and its cos. $= \sqrt{(1 - cczz)}$. Consequently
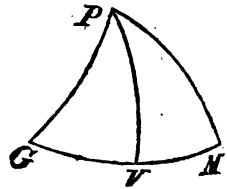
$$ddxx + dfx + ff = 1 - bbzz; \quad zz = \frac{1 - ddxx - 2dfx - ff}{bb};$$

$$ggxx + 2ghx + hh = 1 - cozz; \quad zz = \frac{1 - ggxx - 2ghx - hh}{cc}.$$

And therefore these two are equal to one another. But $dd \div bb = gg \div cc$; so the two terms wherein xx is found destroy each other.

We have therefore $x = \dfrac{cc + bbhh - bb - ccff}{2ccdf - 2bbgh} = \cdot969343$ the co-sine of $14° 13'\, 27''$, and the difference of longitude of Gibraltar from Moscow $28° 26'\, 54''$.

Vienna and Gibraltar bears from Moscow, S. $56° 4'$ westerly.

Moscow from Vienna, north $44° 50'$ easterly.

Gibraltar from Vienna, south $44° 50'$ westerly.

Vienna and Moscow from Gibraltar, north $35° 16'$ easterly.

Vienna is distant from Gibraltar $16° 29' = 1146$ English geom. miles; Vienna from Moscow $11° 23' = 791$ miles; Gibraltar from Moscow $27° 52' = 1937$ miles.

Questions proposed in 1740, and answered in 1741.

I.   QUESTION 213, by Mr. Rob. Heath.

What sum of money x, will double itself in x years at x per cent. per annum, compound interest?

Answered by Mr. J. Turner.

Let $x =$ principal, rate, and time; as $100 : x :: 1 : \cdot01x$. And putting $b = \cdot01$; what Ward in his Comp. Interest calls R, will be $= 1 + bx$; consequently by his 1st prop. and per quest. $x(1+bx)^x = 2x$; or $(1 + bx)^x = 2$. Hence $x \times$ log. $(1 + bx) =$ log. $2, = \cdot693147$ (or c). But the log. of $(1 + bx)$ is $= bx - \dfrac{b^2x^2}{2} + \dfrac{b^3x^3}{3} - \dfrac{b^4x^4}{4} + \dfrac{b^5x^5}{5}$, &c. which multiplied by x is $= c$. Reverted, it gives $x = 8\cdot4983$ the time and rate, with which any sum will gain the principal.

Answered by Mr. Peter Kay.

Let x be the number sought; therefore $\left(1 + \dfrac{x}{100}\right)^x = 2$, per

quest. this in log. gives $x \times \dfrac{x}{100} - \dfrac{x^2}{20000} + \dfrac{x^3}{3000000}$, &c. = ·693147, whence $x = 8\cdot4983$, &c.

II. QUESTION 214, *by Mr.* Nich. Farrer.

Sometime in the spring quarter, in 1739, in the forenoon, an observation being made of the sun, his altitude was found 33° 41′ 40″; and azimuth from the north 102° 40′ 52″; and sometime after, on the same forenoon, his altitude was found 48° 46′ 53″, and azimuth 134° 39′ 56″. From whence the latitude of the place of observation, month, day, and hours of observation may be found and are here required? With a general theorem for all questions of this nature.

Answered by Mr. Ant. Thacker.

Construction. With radius = sine of 90° describe the primitive circle on the plane of the meridian HZHN; draw HH through the centre for the horizon, and ZN the east azimuth at right angles thereto; with the versed sines of 102° 40′ 52″ and 134° 39′ 56″, set off H*t* and HT : And with the sines of 33° 41′ 40″ and 48° 46′ 53″ lay down c*k* and CK; then draw *k*Q and KQ parallel to HH: make N*u* and N*e* = *kq* and KQ; join ⊕N and *t*N; and draw *w*u and *he* parallel to HH; then make *k*⊙ and K⊙ = *wu* and *he*; through ⊙⊙ draw G⊙⊙*mng*, and draw the equator ÆCÆ parallel thereto; and c*np*

the elevation of the pole at right angles to it. From whence *cn* will be the sine of the sun's declination, and P*b* the sine of the latitude, &c.
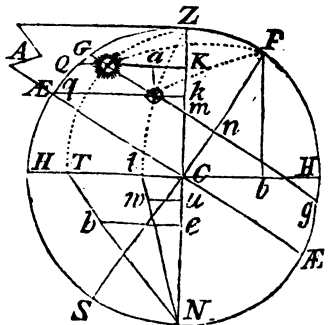
1. From which projection, the investigation of a general theorem will easily arise. For per similar triangles, say as NC (radius = 1) : c*t* (*z*) :: N*u* (*q*) : *qz* = *uw* = *k*⊙. And NC (1) : CT (*m*) :: N*e* (*s*) : *ms* = *he* = K⊙; ∴ *ms* − *qz* = ⊙*a*. Also as ⊙*a* (*s* − *p*) : ⊙*a* (*ms* − *qz*) :: CZ (1) : AZ = (*ms* − *qz*) ÷ (*s* − *p*) = the tang. of the required latitude.

2. *For the Sun's Declination.* As P*b* (*x*) : c*b* (*y*) :: ⊙*k* (*qz*) : *km* = *qzy* ÷ *x*; and as m*c* (*p* − *qzy* ÷ *x*) : *cn* (*d*) :: CP (1) : P*b* (*x*); therefore *px* − *qyz* = *d*, the sun's declination.

3. *Lastly, for the hour of the day.* Put *c* = cosine of sun's de·

clination; and s and e for the sine and cosine of the hour from noon; then will $px - qyz = d$; and $dx + cye = p$;* and by substitution $xxp - xqzy + cye = p$; but $1 - yy = xx$; $\therefore ce = py + xqz$; and as $s : q :: n$ (sine of ☉ azimuth) $: c$; $\therefore qn \div s = c$; which substituted for c, gives $qne \div s = py + xqz$; therefore $e \div s =$ $(py + xqz) \div qn =$ cotangent of the hour from noon, at the first observation.

* *N. B.* If the sun's azimuth is less than 90° (from the north) then tc must be taken on the contrary side of c, and therefore negative with respect to what it is (in this quest.)

Hence $\dfrac{qz - ms}{s - p}$; $px + qyz = d$; and $\dfrac{s}{e} = \dfrac{py - xqz}{qn}$ are general expressions for the quest. as above. Only whenever any angle or side exceeds 90°, its cosine must be expressed by a negative sine.

This theorem $px + qyz = d$, will be found to answer all the ends proposed by that of Mr. Auderson, in the Diary, 1732. And is indeed more simple, and is better adapted to the uses than his; though I freely own, that the above method of deriving it was first hinted to me some years ago by my ingenious friend Mr. T. Simpson.

This theorem, *viz.* $\dfrac{e}{s} = \dfrac{py - xqz}{qn}$ is found to be more useful than the other, e. g. In the solution of the prize question, 1739, solved by it, comes out $st \div 2bc = x$; where the sine $= s =$ GP $+$ MP $=$ sine 89°; and $t =$ tangent PV; and all the others as by Mr. Turner, Diary, 1740. Also quest. 211 may readily be answered by this theorem, and will come out $xx + 2px = n$.

These theorems above being brought out in numbers, by help of the artificial and natural sines, to facilitate the labour; gave, 1. The latitude $= 54°$ 51'. 2. The sun's declination $= 20°$ 24' answering to the 10th day of May. And 3d, the hour $60° = 4^h$ or 8 in the morning, and $30° = 2^h$ or 10 o'clock, agreeing precisely with that of the ingenious proposer's true answer.

III. QUESTION 215, *by Mr.* J. May.

Going to pass a leisure hour at billiards, I wondered to find the table an irregular hexagon, when seeing the balls fly very strangely in striking the several gins, made me think, If two balls A and B, lay on the said table, and the ball A was struck against the gin RS, from thence reversing to ST, from thence to TV, then to VW, then to WX, thence to XR, thence reversed, and struck the ball B; to find geometrically the points in the several gins, where the ball A will strike; and that by a general construction for all polygons, supposing the balls to be geometrical points?

Answered by Merones.

Produce all the sides of the figure as in the scheme: on Rs let fall the perpend. Apa, and make $ap = $ Ap; draw ab ⊥ sᴛ, and make $bg = ga$; draw bc ⊥ ᴛv, and make $ch = bh$. Also Bf ⊥ Rx, and make $qf = $ Bq; draw fe ⊥ wx, and make $ei = fi$; draw ed ⊥ vw, and make $dk = ke$. Then draw the lines cd, $3b$, $2a$, $4e$, $5f$, and 6B, cutting the sides of the figure; then A123456B will be the path of the moving body A.
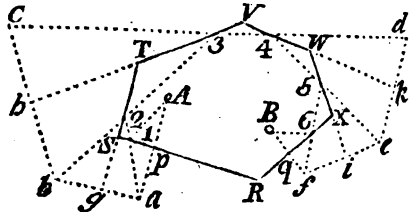
IV.   QUESTION 216, *by Mr.* Henry Travis.

At Matlock, near the Peak in Derbyshire, where there are many surprising curiosities in nature, is a rock by the side of the river Derwent, rising perpendicularly to a wonderful height; which being inaccessible, I endeavoured to measure in a mathematical method. From a station at some distance, (nearly level with the bottom of the rock) I took an angle of altitude to its top 47° 30′; and having designed a second station, I took an horizontal angle 87° 5′, between the foot of the rock and that station; the measured distance between the stations was 4 chains and 29 links, (per Gunter) or 283·274 feet. At that place I had an angle of altitude 40° 12′, but forgot from hence to take an angle between my first station and the foot of the rock; yet am in hopes some curious artist will, from this data, determine the perpendicular height of this stupendous rock?

Answered by Mr. N. Farrer.

In the annexed figure, if Bᴛ, Bᴛ, represent the height of the rock perpendicular to the horizontal plane Aʙc; the points ᴛ and ᴛ (on the turning up the two triangles to which and the plane they are perpendicular) are supposed coincident; A and c the two stations, and ʙAc the horizontal angle given: by letting fall a perp. cᴅ by plain trigonom. I find cᴅ $= m$, and Aᴅ $= p$. Then put $q = $ sine ∠ʙAᴛ, (the angle at the first observation) and $s = $ its cosine; $h = $ sine ∠ʙcᴛ (∠ altit. at the second observation) and $d = $ its cosine.   Let $x = $ Dʙ, then (per 47 Eucl. I.) $\sqrt{(xx + mm)} = $ cʙ and $x + p = $ Aʙ; then as cosine ∠ʙAᴛ : its sine :: Aʙ : Bᴛ; cos. ʙcᴛ : its sine :: ʙc : Bᴛ;
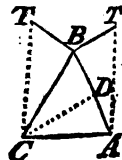
i. e. $s : q :: x + p : \dfrac{qx + qp}{s} =$ BT; $d : h :: \sqrt{(xx + mm)} :$

$\dfrac{h\sqrt{(xx + mm)}}{d} =$ BT.　Then $\dfrac{h\sqrt{(xx + mm)}}{d} = \dfrac{qx + qp}{s}$; hence

$sh \sqrt{(xx + mm)} = dqx + dqp$. This equation squared and brought into numbers, the square completed and root extracted, &c. I find $x = 311\cdot24$ feet; and BT the rock's height $= 118\cdot47$ yards, the answer to the question.

N. B. By a mistake in printing, the angle was 40° 12′ instead of 41° 12′; so the true height was more, as in the following answer.

The same answered by Mr. H. Travis.

Let FSB represent the ground lines, as supposed truly horizon-tal; in which is given FS the stationary line 429 links $= a$; and the horizontal angle SFB $= 87° 5' = b$; and in the vertical $\triangle$ BFR the angle of altitude at F $= 47° 30' = s$; in the other $\triangle$ BSR, right-angled at B, the second angle of altitude BSR $= 41° 12'$
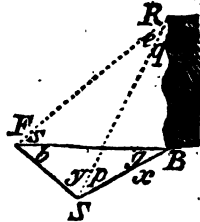
$= p$. Then say, as $q : x :: p :$ BR $= \dfrac{px}{q}$;

as $s : \dfrac{px}{q} :: e :$ BF $= \dfrac{pex}{sq}$; and as $b : x :: y$

$:$ BF $= \dfrac{xpe}{sq}$; $\therefore yx = \dfrac{pebx}{sq}$, and $y = \dfrac{peb}{sq} = \angle$s.

The Operation by Logarithms.

$p = 9\cdot8186807$
$e = 9\cdot8296833$　　$s = 9\cdot8676309$
$b = 9\cdot9994370$　　$q = 9\cdot8764574$

$peb = 29\cdot6478010 - qs = 19\cdot7440883$ gives $9\cdot9037927 = 53°$ 15′, which added to 87° 5′, then subtracted from 180, leaves 39° 40′

$= \angle$B, whose natural sine call g; then as $g : a :: b : x = \dfrac{ba}{g}$.

Which substitute for x in the value of BR gives $\dfrac{bap}{gq} =$ the height of the rock. The operation:

$b = 9\cdot9994370$
$a = 2\cdot6324573$　　$g = 9\cdot8050385$
$p = 9\cdot8186807$　　$q = 9\cdot8764574$

$bap = 22\cdot4505750 - gq = 19\cdot6815959 = 587$ links or 129 yards.

N. B. b, e, g, &c. do not denote the angles s, F, B, &c. but their sines. L.

V. QUESTION 217, by Mr. Ant. Thacker.

Given the * equation of the exponential curve MDSEB, together with the axis AB $= b = 1000$; to find the greatest ordinate (SR) and inscribed parallelogram DEQP, and to give the analytical investigation of the same?

$$* \quad \text{PB}^{\text{AP}} = \text{PD}^{\text{PD}}; \text{ i. e. } (b-x)^x = y^y,$$

Answered by Mr. Rob. Heath.

When an ordinate (or any quantity) is a maximum, the logarithm of it, or any power of it is a maximum;

consequently the log. of $\text{RS}^{\text{RS}} = $ log. of RB^{AR} $= x \times$ log. of $b - x$ is a maximum.
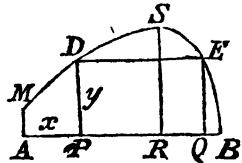
In fluxions $- \dfrac{x\dot{x}}{b-x} + \dot{x} \times$ log. $(b-x)$

$= 0$; where x is easily determined $=$ 836·0532 &c. by a new method of solving exponential equations; ∴ $(163\cdot9468)^{963\cdot053} = y^y$, and $y = 657\cdot1442$ &c. very true to a decimal, the length of the greatest ordinate RS required, By drawing the true figure, or trying the increase of the parallelogram, it appears that AP is greater than 400, when the ▭QEDP $=$ DP $\times$ PQ is a maximum; and by the table of logarithms, it also appears that DP $=$ EQ is greater than 420, and AQ less than 987. This must be observed in order to have the following logarithmetic series converge   Put $400 + x =$ AP, $420 + y =$ DP $=$ EQ, and $987 - z =$ AQ; to find DP and PQ.   By the equation of the curve, $\text{PB}^{\text{AP}} = \text{PD}^{\text{PD}} = \text{QE}^{\text{QE}} = \text{QB}^{\text{AQ}}$, or $(600-x)^{400+x} = (420+y)^{420+y}$ $= (13+z)^{987-z}$, or $(m-x)^{n+x} = (p+y)^{p+y} = (q+z)^{r-z}$ by substitution; in logar.

$$\text{l. } m \overset{\ast}{\times} n - \frac{n}{m} \left.\right) \quad - \frac{n}{2mm} \left.\right) \quad - \frac{n}{3m^3} \left.\right) \quad x^3 \text{ \&c.} = \begin{matrix} \text{l. } p \times (p+1) \\ + \text{l. } p \end{matrix} \Big\} y$$
$$+ \text{l. } m \left.\right) \quad - \frac{1}{m} \left.\right) \quad - \frac{1}{2mm} \left.\right)$$

$$+ \frac{3}{2p}y^2 + \frac{5}{6pp}y^3 \text{ \&c.} = \text{l. } q \times r + \frac{r}{q} \left.\right) \quad + \frac{r}{2qq} \left.\right) \quad + \frac{3}{3q^3} \left.\right)$$
$$- \text{l. } q \left.\right) \quad - \frac{1}{n} \left.\right) \quad + \frac{1}{2q^3} \left.\right) \text{ \&c.}$$

Now the value of any two ingle quantities may be found in terms of the other.   The above in numbers (g) $21\cdot8648 + (a)$ $5\cdot730263x$

— (h) ·0022221x^2 — (k) ·000002x^3 &c. = (c) 7·040254y + (d) ·0035714y^2 + (e) ·0000047y^3 &c. and = (c) 73·358128z + (d) 2·843195z^2 + (e) ·146791z^3 : I find the values of y and z in terms affected with x, by assuming an equation of this form, *viz.* y or z = A + Bx + cx^2 + Dx^3 &c.   Where by substitution for their assumed values, and making the coefficients of like terms on different sides of each equation equal, we shall have the real respective values of the coefficients, *viz.* A^3 + dA^2 + cA = g; A = 3·1008 and ·36508;

$$B = \frac{3eA^2 + 2dA + c}{a} = \text{·811364 and ·075904; } c =$$

$$\frac{— b — 2dB^2 — 3eAB^2}{3eA^2 + 2dA + c} = — \text{·0006478 and — ·0002586; } D =$$

$$\frac{— k — 2dBC — 6eABC — eB^3}{3eA^2 + 2dA + c} = — \text{·0000001005 and + ·000000686.}$$

Consequently y = 3·1008 + ·811364x — ·0006478xx &c. z = ·36508 + ·075904x — ·0002586xx &c. wherefore 586·63492 — 1·075904x + ·0002586xx &c. = PQ, and 423·1008 + ·811364x — ·0002586xx &c. = DP; the product of these two is a maximum, which fluxed and reduced, there results 20·759 — 2·28625x + ·003212xx &c. = 0; here x = 9·2 nearly; whence PQ = 576·758, DP = 430·5105, and the area of the greatest ▭ 248300$\frac{1}{4}$.

Answered by Mr. J. Turner.

The equation of the curve is $(b — x)^x = y^y$; AB=1000=b; hence x × l. $(b—x)=y$ × l.y. In fluxions, $\dot{x}$ × l. $(b—x)$ — $\dfrac{x\dot{x}}{b—x}$ = 0; $(b—x)$ × l. $(b—x)$ = x.   Or finding the logarithm thereof, $(b—x)$ × l. b + $\dfrac{x}{b}$ — $\dfrac{x^2}{2b^2}$ + $\dfrac{x^3}{3b^3}$ — $\dfrac{x^4}{4b^4}$ &c. = x.   This reduced gives x = 836 nearly; whence RS the greatest ordinate is = 657, &c.

For the greatest Parallelogram :

Put z = PQ, DP = EQ = y; the area = A.

By the nature of the curve, $(b — x)^x = y^y$,

and $(b — x — z)^{x + z} = y^y$,

and zy = A.

Which thrown into fluxions and reduced (by expelling z) gives x = 407·6, y = 429·2, z = 578·4, AQ = 986, and the required area 248250.

The same answered by Merones.

1. For the greatest ordinate: $(b — x)^x = y^y$, and x × log.

T 4

$(b - x) = y \times \log. y =$ max. whence $\dot{x} \times l. (b - x) - \dfrac{x \dot{x}}{b - x} = 0$; and $(b - x) \times \log. (b - x) = x.$ Put $a + v = x$; $b - a = g$; and then $(g - v) \times l. (g - v) = a + v.$ Let $p = \log. g$: take $a =$ 836; then $gp - a - pv - 2v + \dfrac{v^2}{2g} + \dfrac{v^3}{6g^2} + $ &c. $= 0.$ And by reversion of series $v = \cdot 0531493$, $x = 836 \cdot 0531493$, and $y =$ 657·02682 the greatest ordinate.

2. Then for the greatest inscribed parallelogram; let AP $= x$, AQ $= s$; then $y \times (s - x) =$ max. put $d + u = s$, $f = b - d$, $r = \log. f$; then $(d + u) \times l. (f + u) = (d + u) \times l. (g - v) (= y \times \log. y)$. These turned into series, and reverted, will give the value of u; whence is had s. Also let $c + z = y$, then will $(a + v) \times \log. (g - v) = (c + z) \times \log. (c + z)$. This turned into series, and the series reverted, finds z, and consequently y; then these values of s and y being substituted in the maximum, and its flux. made $= 0$, and the series reverted, gives v, x, and y, and thence the inscribed parallelogram $= 247675$ nearly.

The same answered by the Proposer, Mr. An. Thacker.

Assume AP $= 407 + x$; PD $=$ EQ $= 429 + y$; and QB $= 13 + v$: then will PQ or DE $= 580 - x - v$; and per nature of the curve we have $(593 - x)^{407 + x} = (429 + y)^{429 + y}$ and $(13 + v)^{977 - v} = (429 + y)^{429 + y}$. These in log. are $(407 + x)$: $\times l. 593 - \dfrac{x}{593}$

&c. $= (429 + y)$: $\times l. 429 + \dfrac{y}{429}$ &c. And $(987 - v) \times : l. 13$

$+ \dfrac{v}{13}$ &c. $= (429 + y) \times : l. 429 + \dfrac{y}{429}$ &c. And these equations ordered make $a + bx + cxx$ &c. $=$ A $+$ BY $+$ Cyy &c. And $f + gv + hvv$ &c. $=$ A $+$ BY $+$ Cyy &c. Consequently $x =$ F $+$ Gy $+$ Hy$\dot{y}$ &c. and $v =$ K $+$ Ly $+$ Myy &c. Hence the area of PDEQ will be expressed by $(o - $ Py $- $ Qyy &c.$) \times (429 + y)$, which per quest. must be a maximum; then put into fluxions and reduced, gives $y = 1 \cdot 5104$ &c. $\therefore$ the greatest area $= 248300 \cdot 3749$ &c. Again, for the greatest ordinate, put $835 + x =$ AR, then is $(165 - x)^{835 + x}$ to be a maximum; therefore put into fluxions and reduced gives $x = 1 \cdot 0531$ &c. and SR $= 657 \cdot 0268$ the true solution.

VI.   QUESTION   218,   *by Mr.* Rich. Gibbons.

I will undertake, with 12 fair dice, to throw 42 once in 15 times; and between 37 and 47, at every throw. Quere, whether I shall be a gainer or loser by these chances, and the exact odds?

Answered by Mr. Peter Kay.

It is found from theorem 2, page 53, of Simpson's Laws of Chance, that the odds between 37 and 47 coming up at any assigned throw, with 12 dice, is as 1·162 &c. to 1, or as 7 to 6 very nearly; which is an answer to one part of the question. And by theorem 1 in the same page, the probability of throwing just 42 comes out $= \dfrac{1}{15\cdot22}$, &c.

Therefore $\left(1 - \dfrac{1}{15\cdot22}\ \&c.\right)^{15} = \cdot3604$ is the probability of losing in the other part of the question; and consequently the required odds for winning as 6396 to 3604, or as 16 to 9 nearly.

The same solved by Mr. J. May, *jun.*

Put the number of dice $= n$; the number of sides on each dice $= m$; $42 = p$; (but if p was greater than $\frac{1}{2}n + \frac{1}{2}mn$, subtract it from $n + nm$, and put the remainder $= p$). Then the chance the proposer has to win in the first throw, or to throw just 42, will be

$$\dfrac{n \times n + 1 \times n + 2 \times n + 3 \times n + 4}{1 \times 2 \times 3 \times 4 \times 5}\ \&c.$$ The number of terms

must be $p - n - \dfrac{n}{1} \times \dfrac{n + 1 \times n + 2}{2 \times 3}$ &c. (in terms less than the

foregoing) $+ \dfrac{n \times n - 1}{1 \times 2} \times \dfrac{n \times n + 1 \times n + 2 \times n + 3 \times n + 4}{1 \times 2 \times 3 \times 4 \times 5}$

&c. (In terms less) $- \dfrac{n \times n - 2 \times n - 3}{1 \times 2 \times 3} \times$

$\dfrac{n \times n + 1 \times n + 2 \times n + 3}{1 \times 2 \times 3 \times 4}$ &c. (in terms less) which put in

numbers is 144840476; now all the chances in the said dice will be m^n or 2176782336; so that there is 2031941860 chances to lose, that is very near 3365 to win, against 47207 to lose the first throw: Put $3365 + \cdot47207 = r$; $47207 = s$; and $15 = t$; the proportion which the proposer has to throw 42 once in 15 times, is

to the chances he has not, as $1 - \dfrac{s^{-t}}{r^{t}}$ to $\dfrac{s^{-t}}{r^{t}}$ (see Struyck p. 52)

which is about 71199 to 128801. The second query is easily deduced from the above series, and the proposer has 1400482914 chances to win, against 776299422 to lose; *i. e.* if 37 and 47 are included; but if excluded, then 1194427194 to win, against 982355142 to lose.

Mr. Rob. Heath *answers this Question.*

The number of chances for 42 happening in one throw with 12 dice (by his theorem) will be 144840476 ; which taken from all the chances on all the dice, leaves 2031941860 chances for failing in one throw.——The advantage in wagering to throw 42, once in 15 times with 12 fair dice, will be 6440045 to 3559955 ; or nearly as 9 to 5.   To find the number of throws to make an equal wager; make $\dfrac{bx}{(a+b)^x} = \frac{1}{2}$, which comes to $x = \dfrac{\log. 2}{1. a + b - 1. b}$ when reduced; and solved $x = 10\cdot0666$ throws by common logarithms:

Mr. De Moivre says $\dfrac{b}{a} \times \cdot7$ shews the trials requisite to that effect, when b is pretty large in respect of a; but in this case it shews it to happen in $9\cdot82$ throws ; very near the truth : but his Table of Limits at p. 42 Doct. Chances, 2d edit. is not very exact, as being not deduced from exponential equations truly solved.

The sum of the chances for each party, N. in one throw (by the series) are found $= 1194307074$; which taken from all the chances on all the dice, leaves 982475262 chances for missing ; therefore the odds are nearly 6 to 5.   *Q. E. F.*

THE PRIZE QUESTION, *by Mr.* Rob. Heath.

It is proposed to enclose a park, the greatest possible, in the following manner : The straight bank of a river, having two trees growing at its extremities, at the distance of three furlongs, is to form one part of the boundary, and the other part is to be a mile in length.   Two equal circular coverts or shades are to be made so as to touch the boundary of the park, and the longest right line that can be drawn from each of the trees to the opposite side of the park.   Required the form and area of the park, and also the area of each of the coverts ?

Answered by Mr. H. Travis.

Put 1 = radius of the circle; TMAIT the park; $q = 1\cdot570796$

&c. ($\frac{1}{4}$ periph. when rad is 1) = quadrantal arch AM ; TM $=x$; TL $=y$; then, as 8 : 3 :: $q+x : y$; $\therefore 8y = 3q + 3x$; and by Sir Isaac Newton's series, $y =$

$$1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} \&c.$$

$$\therefore 8 - 4xx + \frac{x^4}{3} - \frac{x^6}{90} \&c. = 3q + 3x.$$

Solved, $x = \cdot0611928$ &c. $\therefore y = \cdot81849$ &c. and the area of the park $= 89$ a. 0 r. 14 p.   For the area of the shades.   Put $s=$ sine arc TM ; $z = po =$ radius of the circular shades ; then, as $s : z :: 1 : 1 - z$; $\therefore z = \frac{s}{1+s}$; hence the area of each is 14 acres and 7 perches.
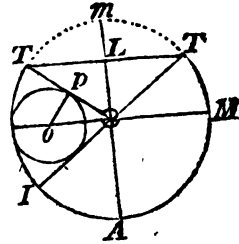
The same answered by Mr. Heath, *the Proposer.*

Since the river TT (part of the boundary) is given straight, there-fore the nearest the park can approach the form of a circle, is the segment TMAIT the true form ; which comprehends for its area a maximum.   Put $b = 320$ poles $=$ TMAIT ; $d = 60$ poles $=$ TLT ; $p = 6\cdot28318$ &c. $x =$ arch mT ; then $(b+2x) \div p =$ rad. $= \odot$T. And, by a series for finding the sine from the radius, and arch given, $d =$

$$x - \frac{x^3}{6 \times \left(\frac{b+2x}{p}\right)^2} + \frac{x^5}{120 \times \left(\frac{b+2x}{p}\right)^4} - \frac{x^7}{540 \times \left(\frac{b+2x}{p}\right)^6} \&c.$$

Where x is found $= 70\cdot287$ &c. whence rad. $= 73\cdot302$ &c. poles $=$ h ; and the area of the park 14255$\cdot$15 poles, or 89 a. 0 r. 15 p.

Put $f = \cdot574459 =$ sine $\angle o \odot p = \angle \odot$TT ; $y =$ radius of each shade $= op$.   Then $h - y = o\odot$; say rad. 1 : $h - y :: f : y$; whence $y = fh \div (1+f) = 26\cdot745$ &c. poles ; consequently the area of each shade $= 14$ a. 0 r. 7 p.

This question may be more easily solved, by a table of nat. sines (with rad. unity).   Seek a chord and arc in the ratio of the given chord and arc, as 3 to 8, which are found by 2 or 3 trials $=$ 1$\cdot$6370516 and 250° 7' 30'' similar to the chord TT, and arc TMAIT ; (for $\cdot8185258$ is sine of 54° 56' 15'' similar to the sine LT, and arc TM, half the supplement of the given arc to 360°) say 1$\cdot$6370516 : rad. 1 :: 120 (TT) : 73$\cdot$302 &c. poles $=$ T$\odot$, radius as found before.

The same solved by Mr. J. Powle.

Put $b = 5400$ the min. in a quad. $c = \cdot000000084616$; $d = 4 =$ $\frac{1}{2}$ arc ; $q = 3\cdot14159$; $p = 1\frac{1}{2} =$ half the given chord ; $x = \angle \odot$TL

in minutes; $y = \text{T} \odot \text{L}$; radius $=1$. It will hold, $b+x: d :: 2b:$

$\dfrac{2bd}{b+x}$ length of the semicircle; $q : 1 :: \dfrac{2bd}{b+x} : \dfrac{2bd}{qb+qx} = \odot \text{T};$

then, per sim. $\triangle s$, $1 : y :: \dfrac{2bd}{qb+qx} : p$; i. e. $\odot a : \text{AB} :: \odot \text{L} : \text{LT};$

$\therefore \dfrac{2bdy}{qb+qx} = p.$ Here $x = \dfrac{2bdy - qpb}{qp} : xx =$

$\dfrac{4bbddyy - 4dbbpqy + qqbbpp}{qqpp}.$ Then by Mr. Ward's equation for
finding the sine corresponding to a given arc (Math. Introd. p. 358)
we have $y^4 + 28y^3 + 195y^2 + 36cxxyy + 108cxxy - 28y = 196 -$
$81cxx$; and exterminating xx, by means of its value before found,
there will be produced this equation:

$$\begin{array}{c}
\dfrac{qqpp}{144cbbdd} y^4 - \begin{array}{c} 28qqpp \\ 144dbbpqc \\ 432cbbdd \end{array} y^3 - \begin{array}{c} 195qqpp \\ 36qqbbppc \\ 432cdbbpq \\ 324cbbdd \end{array} y^2 - \begin{array}{c} 108qqbbppc \\ 28ppqq \\ 324cbbdqp \end{array} y
\end{array}$$

$= 196qqpp - 81qqbbppc.$

In numbers, $-5707 \cdot 1022y^4 - 10979 \cdot 01y^3 + 998 \cdot 257y^2 +$
$9733 \cdot 248y = 85 \cdot 7091935$; solved, $y = \cdot 8185287$ the sine of $54°$
$56'$ $16''$; by trigonom.s. $54°$ $56' : (\text{LT})$ $1 \cdot 5 ::$ rad. $90° : \text{CL} = 1 \cdot 83255;$
then $1 \cdot 8325 \times 4 = 7 \cdot 3302 = 73$ a. 1 r. 8 p. the sector; and
$\sqrt{(3 \cdot 3325 \times \cdot 3325 \times 1 \cdot 5 \times 1 \cdot 5)} = 15$ a. 3 r. 6 p. $=$ area $\triangle$.
Their sum 89 a. 0 r. 14 p. the true area of the park. Shade 14 a.
0 r. 7 p.

Questions proposed in 1741, and answered in 1742.

I. QUESTION 219, *by Mr. J. May, of Amsterdam.*

Here, in Holland, the land lying so very low, they are obliged to
raise banks or dikes to keep the sea from overflowing it; yet some
time ago a great storm, with a high tide, broke through one of the
banks, and laid the country for some miles under water. Going with
some friends to the place where this inundation happened, and walk-
ing upon one of these narrow banks, we saw at a distance three trees
(A, B, c) standing in the water, which we were told stood at an equal
distance from each other, at the corners of a triangular field; and a
pole equidistant from each tree, that formerly was used as a mark to

shoot at with bows and arrows. Now one of our party proposed to find the content of the inundated field by the help of a staff, or measuring rod only, which we had with us. As we walked along this bank in a straight line (DP) at D we came in a straight line with the trees c and A; thence measuring 304 feet, at E we came in a right line with the trees c and B; thence continuing straight 121·6 feet, at R we made a right line with the pole in the middle of the field, and the tree at the farther corner B; lastly, from R measuring right forward 159·6 feet, at P we came in the line of BA. From this, which was all we were then capable of doing, is required the content of the field ABC?

Answered by Mr. R. Fall, of Dunbar.

As PR + RE (281·2) is to PR — RE (38), so is the tang. of half the sum of the (opposite) angles P and E; to tangent of half their difference (13° 10'); and half the sum more ½ the difference is equal to the greater angle (PEB = 73° 10'); the half sum (60) less ½ difference is the lesser angle (EPB = 46° 50'.)
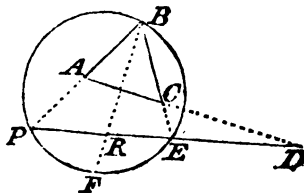
Then say, as sine ∠B (60°) : PE (281·2) :: s. P (46° 50') : BE (236·9). Again, s. ∠ECD (60) : ED (304) :: s. ∠D (13·10) : EC (79·959); ∴ BE — EC = 156·94 = BC the side of the triangle required. Which squared and multiplied by $\sqrt{\frac{3}{4}}$ give 10664·857 square feet, the area of the triangle ABC required.

There is a great many ways this problem may be solved, which I I shall reserve for another place, where the solutions to the questions in the Diaries shall be further discussed and illustrated. Only here observe, that by 3 Eucl. 6, the line BLR bisecting the ∠B, cuts the opposite side into two segments PR and RE, in proportion to the other two sides (BP and PE), and consequently the sines of the angles they subtend in like ratio.

Additional Solution.

Construction. Upon PE describe the segment of a circle capable of containing an ∠of 60°; and having completed the circle, from P, the middle of the lower segment, through R draw a line to cut the circumference in B; join the points P, B, and E, B, and ⊥ FB draw DL, cutting EB and PB in c and A, and ABC will be the triangle whose area is required.——For the ∠ABC being, by construction = 60°, the ∠FBE =the half of it or 30°, and the ∠DLB a right one, the ∠BCL will also be = 60°, and of course the △ABC equilateral.   H.

N. B. The letter L is wanting in the figure at the intersection of the lines AC, BR.

II. QUESTION 220, *by* Mcroncs.

Being at sea on the first of May, and a clear forenoon, I made two observations of the sun, and found the difference of altitudes 16° 30′, the difference of azimuths 34 deg. and the difference of the times 2ʰ 15ᵐ. Required the latitude of the place and hours of the day?

Answered by Mr. Ant. Thacker.

As sine 90 : cosine sun's declin. (71° 47′) :: s. diff. of time (2ʰ 15′) : 32° a fourth number = ⊙o the distance in the parallel of declin. Then to the nat. cos. of the diff. of azimuths (34°=·829037) add radius, the sum is 1·829037, which multiplied by the nat. cos. 16° 31′ = ·958819) the difference of altitudes gives 1·753717282920, which made less by twice the cosine (32°=2 × ·848048)=1·696096, leaves ·057621082920; which divided by the versed sine of the difference of azimuths (34° = ·170963) quotes ·337453; which answers to the nat. sine of 70° 18′; one half of which 35° 9′ made less by half the given difference of altit. (8° 15′) leaves 26° 54′ the altititude of the sun at the first observation, and more by 8° 15′ is = 43° 24′ the altit. at the second observation. Whence there is now sufficient given to find, by the common canons in trigonometry, the latitude = 57° 5′, and that the time of the first observation was at 7 26′, and the second at 9ʰ 41′.

The analytical Investigation of the abovesaid Theorem.

Given ⊙P = 71° 47′ comp. declination, the ∠ ⊙PO = 33° 45′ diff. time, whence ⊙o is found = 32°, whose cosine call *m*, put *z* = cosine ⊙zo the diff. azim. *a* and *b* for sine and cosine of ½ diff. alt. then will *bb — aa* or 1 — 2*aa*, or 2*bb* — 1 (= *n*) be the cosine of the diff. of the altitudes. Also let *x* and *y* express the sine and cosine of half the sum of the altitudes: then *yy — xx* (= *v*) will stand for the sum, *xb + ya* and *yb — ax* the sine and cosine of the greater altitude, *xb — ya* and *yb + ax* the sine and cosine of the lesser. Then by a theorem in page 176 of Simpson's fluxions, *xxbb — aayy + yybb — aaxxz* = *m*; or writing 1—*xx* for *yy*, and reducing the equation we have $xx = \dfrac{m + aa - bbz}{1 - z}$;
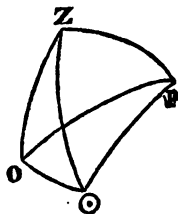
∴ $1 - xx = \dfrac{1 - z - m - aa + bbz}{1 - z} = yy$, and $\dfrac{1 - 2aa - z + 2bbz - 2m}{1 - z}$

$= yy - xx = \dfrac{1 - 2aa + (2bb - 1) \times z - 2m}{1 - z} = \dfrac{n + nz - 2m}{1 - z}$

$$= \frac{n(1+z)-2m}{1-z} = v \text{ the cosine of the sum of the altitudes. In}$$

numbers $\dfrac{\cdot 9588\,(1 + \cdot 8290) - 1\cdot 6960}{1 - \cdot 8290} = \cdot 3375$, which answers in

the tables to 70° 16′ for the sum of the sun's altitudes, whose half (35° 8′) added to the half diff. (8° 15′) gives (43° 23′) the greater altitude; whence the lesser is = 26° 53′ and the latitude = 57° 7′. And the times of observation 7ʰ 27′ and 9ʰ 42′.

III.   QUESTION 221, by Mr. Peter Kay.

One with six dice undertakes to bring up four faces of a sort at a throw; that is, either four aces, four duces, &c. in seven trials? What is the odds against him?

Answered by Mr. Farrer.

First (per p. 7 Simpson's Laws of Chance) $\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times$

$\frac{5}{6} \times \frac{5}{6} \times 6 = \frac{3125}{7776} = $ the probability that one face of a sort comes

up at one throw with 6 dice. Then $\dfrac{3125^4}{7776^4}$ is the probability of four

faces of a sort coming up at the first throw with six dice; and that

of the contrary $\dfrac{7776^4 - 3125^4}{7776^4}$, and the odds $7776^4 - 3125^4$ to

3125^4, or as B to A. Then by page 13, $\dfrac{AP \times B^{n-p}}{(A+B)^n} \times \dfrac{n}{1} \times \dfrac{n-1}{2}$

$\times \dfrac{n-2}{3}$, &c. to p, factors : here $n = 7$, and $p = 1$; which substi-

tute in the above theorem, and it becomes $\dfrac{(3125^4 \times 7776^4 - 3125^4)^6}{7776^7}$

$\times 7 = $ the required probability $= \cdot 8134$, and that on the contrary $\cdot 1866$, and the odds 8134 to 1866, or as 4067 to 933, i. e. 2·1954 to 1.

Answered by Mr. Rob. Heath.

Raise the binomial a (1) + b (5) to the 6th power, the three first terms of which $(a^6 + 6a^5b + 15a^4b^2)$ will be the chances for 6, 5, and 4 aces to come up, which (in this case) being multiplied by 6, = 2436 chances for 6, 5, and 4 aces, duces, trays, &c. to come up

at one throw; wherefore $\dfrac{44220^7}{46656^7}$ are the chances for failing to throw

6, 5, and 4 like faces in 7 trials ; whence the wager's disadvantage is 687033 to 312967, or 11 to 5 nearly, viz. 2·197 to 1.

The same answered by Merones.

The number of combinations of 4 aces out of 6 is 15 ; and in any one case of these 15, any two left out are capable of 25 variations where no ace is found ; therefore 375 gives all the cases where only 4 aces can be cast. But since 5 or 6 aces must be cast, the number of chances for these, which is 31, must be added ; therefore all the cases wherein 4 aces can be cast with 6 dice, is 406. Now since there is the same variety for duces, trays, &c. therefore 2436 is the whole number of cases wherein 4 points of any one sort can be cast : Let the whole number of chances $46656 = s$; $46656 - 2436$ $= q$; the odds against the thrower for 7 casts will be as $\frac{q^7}{s^7}$ to 1 $-$ $\frac{q^7}{s^7}$, or as 2·1954 to 1.

IV.   QUESTION 222, *by Mr.* Daniel Boote.

Some time in the spring quarter 1740, an observation was made of the sun, at 24 minutes past eight o'clock in the morning, and his altitude found 39° 7′. Also at 15 minutes past ten o'clock (on the same forenoon) the altitude was 55° 57′. From whence the latitude of the place of observation, month, and day may be found ; and are required, with a general theorem for all questions of this nature.

Answered by Mr. J. May.

Let a be $=$ sine of sun's alt. at the first observation ; $b =$ that of the second ; $c =$ sine of the hour angle from 6 o'clock at the first observation ; $d =$ that at the second ; and let $r=$ radius ; $b - a = p$; $d - c = q$; $r - c = h$; and $r + d = k$; then the sine of the sun's southern altitude will be $a + \frac{hp}{q}$, the degrees of which put $= m$: Likewise the sine of the sun's depression under the horizon in the north will be $\frac{kp}{q} - b$, whose degrees let be $= n$; then the sun's declination will be $\frac{m - n}{2} = 20° 25′$ N. answering to the 11th day of May ; and the latitude is $90° - \frac{1}{2}m - \frac{1}{2}n = 46° 58′$ north.

This question was answered by Mr. *Boote,* the proposer, and several others. But that we might shew what may be expected from the intended treatise of the Diary questions, &c. where this and other questions will be solved by simple equations, which have their origin

from the application of algebra to spherics, geometry, and all other branches of the mathematics, I will here give you the analytical investigation of a theorem or two for solving this 222d question.

Given $z \odot = 50° \, 53'$, its cos. $= b$; $zo' = 34° \, 3'$, its cos. $= a$; the $\angle \odot \text{PZ} = 54° \, 0'$, its cos. $= c$; $o\text{PZ} = 26° \, 15'$, its cos. $= d$. Put x and y for sine and cosine of half the sum of the arcs $\odot$P and zP, z and v for sine and cosine of half their difference; then will $xv + yz$ and $yv - xz$ be the sine and cosine of the greater arc $\odot$P; $xv - yz$ and $yv + xz$ those of the lesser zP. And (by p. 176, Simpson's fluxions) $yyvv - xxzz + dxxvv - dzzyy = d$. And $yyvv - xxzz + cxxvv - czzyy = b$; and substituting $1 - zz = vv$, and reducing the equations we have $yy - zz + dxx - dzz = a$; and $yy - zz + cxx - czz = b$; and substituting $1 - xx$ for yy, we get $vv + zz = \dfrac{a - ca - b + db}{d - c} = \cdot 89449$ the cosine of
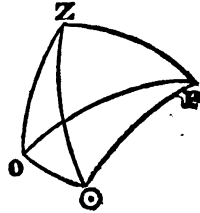
the difference of the arcs zP and $\odot$P ($= 26° \, 34'$) or the sine of the sun's meridian altitude, which answers to $63° \, 26'$. Again, per equation above, we have $\dfrac{b + db - a - ca}{d - c} = - \cdot 38410, \, = yy - xx$, the cosine of the sum of those arcs ($112° \, 36'$) or the sine of the sun's depression at midnight, $= 22° \, 36'$. Whence $90° - \dfrac{63° \, 26' + 22° \, 36'}{2}$ $=$ latitude $46° \, 59'$; and $\dfrac{63° \, 26' - 22° \, 36'}{2} = 20° \, 25'$ the sun's declination on the 11th day of May.

Theorem. $\dfrac{\text{s. } 90° - \cos. 54° \times \text{s. } 55° \, 57' - \text{s. } 39° \, 7'}{\cos. 26° \, 15' - \cos. 54°} + \text{s. } 39° \, 7' = \text{s.}$

sun's alt. at noon. $\dfrac{\text{s. } 90° + \cos. 54° \times \text{s. } 55° \, 57' - \text{s. } 39° \, 7'}{\cos. 26° \, 15' - \cos. 54°} -$

s. $39° \, 7' = \text{s.}$ sun's depression at midnight. Thence the meridian altitude is had $63° \, 34'$; and the midnight depression $22° \, 34'$, half the difference of which is the sun's declination; half of each subtracted from $90°$ is the latitude required.

v. QUESTION 223, by Mr. Peter Kay.

Supposing a body in $51° \, 32'$ north latitude, to be projected with a velocity of six miles per second, in a south-east direction, and an angle of $30°$ with the horizon: Required the trajectory, the place of the earth where the body will fall, and the time it is in motion; allowing the earth to be spherical, its circumference $= 25020$ miles, that a heavy body descends $16\cdot1$ feet in a second of time, and that the projectile moves in a non-resisting medium?

Answered by the Proposer Mr. P. Kay, and by Merones.

Since the velocity with which the ball is projected is sufficient to carry it over a space of 6 miles per second; and the velocity of the place of projection through the earth's rotation round its axis is 0·1802 miles per second; the absolute velocity compounded of these two will be 6·1115 miles per second; and the angle which the true direction of the ball makes with the horizon 29° 24′, and the azimuth or bearing from the south 46° 22′. Let PQRO represent the earth, R the place of projection, ACHB the required trajectory, AOH its transverse axis, CB is conjugate, and RS a tangent to it at the point R. Putting OR = *d*; 6·1115 = *v*; nat. sine of ORS (119° 24′) = *s*; radius = 1; and the distance descended in one second, in parts of a mile, = *r*. Then by p. 23, of Mr. Simpson's Mathematical Essays, the transverse AH will be $= d \div \left(1 - \dfrac{vv}{4rd}\right)$
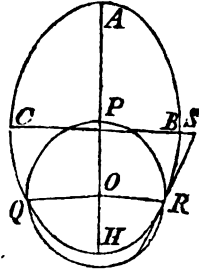
$= 17165$ miles, the conjugate $CB = \dfrac{vs\sqrt{AH}}{\sqrt{r}} = 12633$ miles; and the time of one entire revolution $= 4^h\ 27^m\ 13^s$ from whence the time that the ball is in motion will be found $4^h\ 6^m\ 24^s$ and the arch are described, in the plane of a great circle round the centre of the earth, in that time will be 209° 14′; but the angle, which that circle makes with the meridian, is found above to be 46° 22′; from which angle, and the two given sides including it, the third side of the triangle, or the latitude of the place where the ball descends, may be found, and comes out 28° 16′ south : Also the angle at the pole, from the solution of the same spherical triangle, will be 156° 30′, which being added to 61° 42′ the arch described by the earth about its axis during the time the ball is in motion, gives 218° 13′ westerly, for the difference of longitude required.

Merones's *Answer.*

To answer every part of this problem at large, would require too much room : I shall therefore only explain the method of calculation :

1. Compounding the earth's motion, with the projectile's motion, I find its true velocity to be 6·11228, at an angle of elevation (above the horizon supposed at rest) 29° 23½′, making an angle with the meridian 46° 22½′.

2. By propositions 15 and 17 Princip. I. the latus of the orbit is $=$ 9301·53; the transverse axis of the ellipsis $= 17252·42$; and the periodic time $= 4^h\ 29^m$.

3. By measuring the area of the ellipsis, and part cut off by the

earth's radius; the time of the flight above the earth will be found $4^h 8^m 32·4^s$ in which time it comprehends an arch of the earth (between its rising and falling) $= 208° 37'$.

4. Therefore, by spherical trigonometry, the body falls in south latitude 28° 48'; and the difference of longitude east is 226° 29', from the point of projection supposed at rest.

5. But since the earth's motion in the time of the flight, transfers the place of projection 62° 23' eastwards; therefore the place the body falls in will be in 164° 6' east longitude from the place of projection.

<center>VI.   QUESTION 224, by Mr. Nich. Farrer.</center>

Two ports bear north and south of each other, and distant 50 leagues: A ship sails from the south port to gain the other, and after running a certain distance upon the larboard tack, alters her course, and reaches the north port. On comparing the distances run, it was found that the distance upon the first course exceeded that upon the other by 10 leagues; and the space included by the distance of the ports and the distances run was 796 square leagues. Required the two courses, with the distance run upon each?

Now let us suppose the ship to have sailed uniformly at the rate of 4 miles an hour, and at the same time that she sailed from the south port, another ship sails from the north port, directly south, at the rate of $5\frac{1}{2}$ miles an hour, till she arrives at a port under the same meridian with the former ports, and is then known to be at the nearest distance from the first ship that she could possibly have been during the whole passage, if she had continued her course to the south port. Required the bearing and distance of the middlemost port from the first ship when she brought her starboard tacks on board?

<center>*Answered by Mr. W. Daniel.*</center>

From an analytical process (which shall be published elsewhere) is deduced this theorem. The sq.

root of $\dfrac{796^2}{25^2 - 5^2} + 25^2$; that is,

$\sqrt{(1681·026,} = 41$ nearly, made more by (5) half the compared distances, makes 46 leagues, the distance between the south and middle port; and less by (5) half the compared distance $= 36$, between the middle and north port: therefore 82 leagues is the distance run. The area 796 ÷ by 25 gives 31·84= PE the perpendicular. Which squared and taken from the square of SE,

<center>U 2</center>

leaves 1102·2144, whose root is 33·2 $=$ sp. Let b and a repre-
sent two cotemporary positions of the ships; and their motion at s
to that at p; as 4 to $5\frac{1}{2}$, or as 1 to 1·375, and we have this theorem:

$$\text{sb} = \frac{\text{sn} \times (1·375\text{se} + \text{se})}{1·375^2\text{se} + 2 \times 1·375\text{sa} + \text{se}} = 21·95. \quad \text{And as } 1 : 1·375$$

$::$ 21·95 : 30·18 $=$ sp, which taken from sn, leaves 19·82 $=$ pn.
Also sp $-$ sa leaves 13·38 $-$ pa. And by 47 Euc. I. ae $=$ 34·11
leagues distance of the first ship from the middle port, when she brings
her starboard tacks on board. Then by trigonometry, $\angle$ nse$=43°$ 47'.
Therefore the course of the ship s to e was N.E. 1° 13' northerly;
and the $\angle$ pen $= 27°$ 51' : $\therefore$ the ship bears from d to n, W.N.W.
5° 21' northerly. Lastly, the $\angle$ aep $= 21°$ 53'. Hence the port a
bears from e, W.S.W. 0° 37' westerly.

The proposer, Mr. Farrer's, answer.

Let $2m =$ sn $=$ 50 leagues; $2n =$ se $-$ en $= 10$; $2y =$ se $+$ en
sought. Then $m + y = \frac{1}{2}$ sum of the 3 sides, $n + y =$ se; $y - n$
$=$ en; thence having the 3 sides the square of the area will be $(m+y)$
$(y - m)$ $(m - n)$ $(m + n)$; that is, $y^2 - m^2 = \dfrac{A^2}{m^2 - n^2}$; ergo,

$$y = \sqrt{\left(\frac{A^2}{m^2 - n^2} + m^2\right)} = 41 \text{ leagues}; \text{ and } s\text{e} = 46 \text{ leagues, and en} =$$

36. Hence the course on the larboard tack is N. 43° 48' E. and the dis-
tance 46 leagues; and her course on the starboard N. 62° 11' W. and
distance 36 leagues.

Now, let a and b represent the two ships at their nearest ap-
proach, and let fall the perpendicular bd; and put $q =$ sine $\angle$s;
$p =$ its cosine: $s = 5·5$; $r = 4$ miles; $d =$ sn; and $z =$ an

sought. Then $d - z =$ as; and (per quest.) $s : r :: z : \dfrac{rz}{s} =$ sb;

and (by trig.) $1 : \dfrac{rz}{s} :: p : \dfrac{prz}{s} =$ sd; and $1 : \dfrac{rz}{s} :: q : \dfrac{qrz}{s} =$

db; then sa $-$ sd $=$ ad $= \dfrac{sd - sz - prz}{s}$; substitute $h = 1 + \dfrac{pr}{s}$;

then ad $= d - hz$: And (by 47 Euc. I.) ad^2 + db^2 = ab^2 =

$\dfrac{qqrrzz}{ss} + dd - 2dhz + hhzz$, which must be a minimum, and its

fluxion $\dfrac{2qqrrz\dot{z}}{ss} - 2dh\dot{z} + 2hhz\dot{z} = 0$. This reduced gives $z =$

$\dfrac{dhzz}{qqrrhhss} = 29·5698$ leagues $=$ an; and sa $= 20·4320$ leagues;

sb $= 21·5053$ leagues. And the middle port bears from the first ship,
when she brings her starboard tacks on board, south 66° 52' wester-
ly, and distant 34·62 leagues.

Additional Solution.

Construction. Since the perpendicular of the △ NE**s** is given, be‐ing a 3d proportional to $\frac{1}{4}$NS and the side of the square expressing the given area, the triangle itself may be constructed by prob. 76 of Simpson's Algebra, 2d edit. Or, perhaps, more elegantly thus : At N ⊥ to NS erect NG = the given perpendicular, and continue it to H so that GH = NG ; with centre s and radius SK = SE — NE describe a circle ; then by prob. 12 of Mr. Lawson's Tangencies, find the the centre of a circle which shall touch that already described, and likewise pass through the points N and H, and that centre will be the vértex of the triangle.——By either of these methods the △ being de‐scribed ; on ES, produced and parallel to NS, take SC and CI propor‐tional to the uniform velocities with which the ships move from s and N ; then on an indefinite line drawn through the points I, s, let fall the ⊥ NF, and complete the parallelogram NFBA, so will BA be the minimum sought.—For constituting any other parallelogram N*fba* ; then since, by sim. △ s, s*b* : *bf* or (Euc. I. 34.) N*a* :: SC : CI, and likewise SB : BF = NA :: SC : CI, the points *b* and *a*, B and A will be contemporaneous positions of the two ships ; and by Euc. I. 18, N*f* or *ba* is greater that NF or BA.

Had BA been required of a given length, NF need only have been taken equal to that length, and, it is manifest, the remainder of the construction would have been performed as above. H.

VII. QUESTION 225, *by Mr.* J. Corbett.

In order to survey and divide the triangular field ABC, I bid my men measure the three sides round, whilst I took the angles. The owner would have it in three equal parts nearly divided, but as there was no water but one spring near the middle, in order to have the three fences run straight from it to the three corners, I took the angles there, as well as those at the spring to each corner, as in the scheme are set down ; for he would have the content of each part separate. Having thus much, as angles, sides, &c. I thought surely I might from hence find the lines to the corners (and areas) as well as Mr. Beighton, in his survey of Warwickshire, could find the situation of Newbold, and distance to High Cross, Tripontium, Eathorpe, &c. as he sets forth in the compartment of his map ; for that besides angles I should have three sides measured, whereas he had but one and a piece. As soon as we had done, night approaching, I repaired home : where I found they, blundering, had taken the three lines all in one sum just 100 chains. And I, being ambitious to match this map‐maker, have been puzzling to find a theorem less complex than his ;

Now if any of you could assist me to dò this by a simple equation, I doubt not but his *F. R. S.* might be torn away from his name, and more deservedly put to mine. The angle at A= 50 deg. at B=60°; and c = 70°. Angles from the spring to A and c, 120°; A, B, 130°; c, B, 110°.
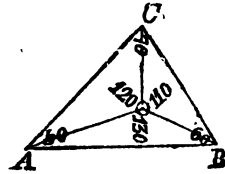
Answered by Mr. John Watts.

The sum of the three sides (per quest.) $= 100$,

The angles $\begin{cases} \text{CAB} = 50° \\ \text{ABC} = 60 \\ \text{ACB} = 70 \end{cases}$ and from the point $\odot$ within $\begin{cases} \text{C}\odot\text{B} = 110° \\ \text{C}\odot\text{A} = 120 \\ \text{A}\odot\text{B} = 130 \end{cases}$

Call CB unity (1); then per trig. as s. $\angle A : 1 :: s. \angle C : AB =$ 1·2266, and s. $\angle A : 1 :: s. \angle B : AC = 1·1305$. And the sum of these three (proportional) sides is $= 3·3671$. Whence by ratio, or proportion,

As $3·3671 : 1 :: 100 : 29·786 \quad = BC$ ⎫ Let x and m express
$3·3671 : 1·2266 :: 100 : 36·538 = BA$ ⎬ the number of degrees
$3·3671 : 1·1305 :: 100 : 33·674 = AC$ ⎭ in the angles AC$\odot$ and
BC$\odot$ respectively"; n and y, those of BA$\odot$ and CA$\odot$; v and s, the angles AB$\odot$ and CB$\odot$. Then $x + m = 70°$, per quest. And $m + 110 + s = 180$ (by 17 Euc. I.) ∴ $m + s = 70°$; consequently x is $= s$. Again, $y + n = 50$; and $n + 130 + v = 180$; ∴ $n + v = 50$; consequently $v = y$. Whence if x and y denote the sine and cosine of the $\angle$CB$\odot$; c and m that of ABC; then will $cy - xm$ express the sine of the angle AB$\odot$ or CA$\odot$. And (per trig.) s. $\angle c\odot$B $(n) : CB (k) :: s. \angle$CB$\odot$ $(x) : \dfrac{xk}{n} = \odot c$; s. $\angle A \odot c$ $(c) : CA (p) :: s.$ CA$\odot$ $(cy - xm) : \dfrac{pcy - pxm}{c} = \odot c$. Consequently $\dfrac{pcy - pxm}{c} = \dfrac{xk}{n}$; which, out of fractions, is $pcny - pxmn = xkc$; ∴ $pcny = xkc + pxmn$; and ∴ $\dfrac{y}{x} = \dfrac{kc + pmn}{pcn} = \dfrac{k}{pn} + \dfrac{m}{c}$, the cotangent of $\angle$CB$\odot$. And $\dfrac{y}{x} = 1·51866$, &c. answering in the tables to $33°$ $21'$. Therefore the angle AB$\odot$ $= 26° 41'$; whence (per trig.) is found,

	Chains.	Poles.				a. r. p.
$\odot$A =	21·383	= 85·5283	⎫		A$\odot$B =	15 1 38·96
$\odot$B =	18·917	= 75·6709	⎬ The area of	⎧	B$\odot$C =	15 1 39·19
$\odot$C =	17·431	= 69·7242	⎭	⎩	C$\odot$A =	16 0 22·31

And the content of the whole field ABC $= 47 \; 0 \; 20·46$

Mr. John Jackson's *answer*.

Calling AC (a); AB (b); BC (c). And finding the sides as above. Putting $m = $ s. A⊙C = CBA, and $d = $ cosine; $n = $ s. ∠C⊙B; and $s = $ s. A⊙B. Then $y = $ s. ∠B⊙A = CA⊙; $\sqrt{(1 - y^2)} = z = $ cosine; and tang. $= \sqrt{\dfrac{1 - y^2}{y^2}} = q$; $x = $ s. CB⊙ = AC⊙. Then

$n : c :: x : \dfrac{cx}{n} = $ c ⊙; and $m : a :: y : \dfrac{ay}{m} = $ c⊙ ; consequently $\dfrac{cx}{n}$

$= \dfrac{ay}{m}$; $\therefore x = \dfrac{any}{cm}$, and $mz - dy = \dfrac{any}{cm}$; transposed, $mz = $

$\dfrac{any}{cm} + dy = \dfrac{any + cmdy}{cm}$, and $z = \dfrac{any + dcmy}{cm^2}$, $\dfrac{z}{y} = \dfrac{an + dcm}{cm^2}$

$= \sqrt{\dfrac{1 - y^2}{y^2}}$; consequently $q = \dfrac{an + dcm}{cm^2} = 1 \cdot 999 = 63° \ 25'$,

$\therefore$ B⊙A $= 26° \ 35'$. Whence the lines and areas as above.

Additional Solution.

The △ ABC itself may be determined by the construction given to prob. 44, in the *Mathematician*, or by first describing a triangle similar to it; after which, if on AB and AC two segments of circles capable of containing ∠s = to 130° and 120° be described, their point of intersection will, it is plain, give the place of the spring. H.

THE PRIZE QUESTION, *by Mr.* J. Turner.

DVF and CVF are two different Apollonian parabolas; V the vertex, and VB the trans. verse diameter of both; DB and AS, ordinates rightly applied: There is given the area DFV equal to $1473\frac{1}{2}$ poles, and VF = FB; F being the focus of the parabola DVB.
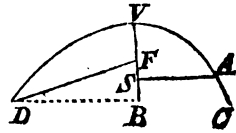
Again, VAC is the other Apollonian parabola, whose arch AV, is to the arch DV, as 1 to 2. And the ordinate AS (rightly applied) being let fall on the common axis VB, makes the area ASV a maximum. Required its parameter, and the area of the wood in statute measure by a simple equation?

Answered by Mr. Ant. Thacker.

Putting $x = $ FV the focal distance, and proceeding with the nature of the parabola, we get $x = 25 = $ FV, $\therefore$ VB $= 50$, and DB $= 70 \cdot 71$.

‡ 4

For any variable abscissa put x; and y for its corresponding semi-ordinate; p for the latus rectum; then will $px = yy$ express the nature of the curve DV; $\therefore \sqrt{(\dot{x}^2 + \dot{y}^2)} = \dfrac{\dot{y}}{p} \sqrt{(pp + 4yy)} =$ the fluxion of the arc DV; whose fluent is $\dfrac{y}{4p} \sqrt{(pp + 4yy)} + \dfrac{p}{4} \times$ l. $\dfrac{2y + \sqrt{(pp + 4yy)}}{p} =$ the length of the variable arc, which when $y = 70{\cdot}71$ is $89{\cdot}8914 =$ DV, the half of which $44{\cdot}9457$ is $=$ the arc VA; which call z, and put $\dfrac{2y + \sqrt{(pp + 4yy)}}{p} = h$; then will $\dfrac{h^4 p - p}{hh} + 4p$ l. $h = 16z$; and $\dfrac{(hh-1)^3}{h^3} pp = 64yx$; or if for p be put a^3, then (1st.) $\dfrac{h^4 a^3 - a^3}{h^2} + 4a^3$ l. $h = 16z$; and (2d) $\dfrac{hhaa - aa}{h}$ $= 4\sqrt[3]{yx}$; whence per (1) equation $\dot{a} = \dfrac{2ha + 4ahh^2 + 2hh^4 a}{3h - 3h^5 - 12h^3 \text{ l.} h}$; and (2d) $\dot{a} = \dfrac{ha + hh^2 a}{2h - 2h^3}$; consequently $\dfrac{2 + 4h^2 + 2h^4}{3 - 3h^4 - 12h^4 \text{ l.} h} = \dfrac{1 + hh}{2 - 2hh}$; hence $h^6 + h^4 - 12h^4$ l. $h - 12h^2$ l. $h - 1 = 0$; which in fluxions, and each term divided by $\dot{h}$, gives $6h^5 - 8h^3 - 4h^3$ l. $h - 24h$ l. $h - 14h =$ A. And proceeding according to case 1st, p. 81 of Simpson's Essays, we get (if no mistake be made) $h = 4{\cdot}133926$; and from the equation $\dfrac{h^4 p - p}{hh} + 4p$ l. $h = 16z$, is had $p = \dfrac{16hhz}{h^4 - 1 + 4hh \text{ l.} h} = 31{\cdot}6$, &c. And the area VSA $= 16$ nearly.

Merones's *answer is thus :*

In the parabola DV we have $2\text{VB}^2 = \text{DB}^2$, and $\frac{2}{3}$ VB $\times$ DB $= 1473\frac{1}{3}$; whence DB$^2 = 4999{\cdot}92$; VB$^2 = 2499{\cdot}96$.

In any parabola, where $x =$ abscissa, $y =$ ordinate, $a =$ latus rectum; the flux. curve $= \dot{y} \sqrt{\left(1 + \dfrac{4yy}{aa}\right)}$ and the curve $= \sqrt{\left(\frac{1}{4}yy + \dfrac{y^4}{aa}\right)} + \frac{1}{4}a \times$ h. l. $\dfrac{2y}{a} + \sqrt{\left(1 + \dfrac{4yy}{aa}\right)}$: therefore VD $= 89{\cdot}8918$, and VA $= 44{\cdot}9459$. Let a parabola be such that $xy = 1$, and the curve a minimum; expunge a out of the foregoing expression of the curve, and put $z =$ h. l. of $\dfrac{2y}{a} + \sqrt{\left(1 + \dfrac{4yy}{aa}\right)}$; and then

$\sqrt{(\frac{1}{4}yy + \frac{1}{yy})} + \frac{y^3 z}{4}$ is a minimum; the fluxion of this made $= 0$,

there comes out $z = \frac{2}{3yy}\sqrt{(1 + \frac{4}{yy})}$; or $\frac{2xx}{3}\sqrt{(1 + 4x^4)} = $ h. l.

$2xx + \sqrt{(1+4x^4)}$; whence by reversion of series, $x = \cdot986408$; $y = 1\cdot01378$; the curve $= 1\cdot4787$; and in the parabola AV, where the curve is $44\cdot9459$; we shall have vs $= 29\cdot9823$; sA $= 30\cdot8142$; and the area vAS $= 615\cdot921$.

This question has exercised the faculties of a great number of persons versed in the most abstruse and higher parts of the mathematics, has occasioned a good deal of speculation and controversy; and the best of artists have been doubtful whether it is possible to be solved by any scientific method; nor can I apprehend I have received any such solutions, unless these two above.

The latter, by Merones, a person so profound in these sciences, that he is equal to the most arduous task; by his difference between the *absciss* and *ordinate* seems to be right, yet do not readily enough comprehend his process: And as I have long wished I could discover who he is, or how to direct to him; I would now heartily beg that favour, that he would please a little further to exemplify how the expression above, $\sqrt{(\frac{1}{4}yy + \frac{1}{yy})} + \frac{y^3 z}{4}$ put into fluxions makes $z = \frac{2xx}{3}\sqrt{(1 + 4x^4)}$.

Remark. The Diary Author having, as above expressed, a desire of having some parts of the solution by Merones (Mr. Emerson) explained to him, and it no where appearing, that I know of, that he received any farther account of it; I think it my *duty*, as an Editor, to supply a particular explanation of the said solution, notwithstanding the Editor of the Repository has thought fit to pass this question by in silence; though his declared intention at the beginning was to ' supply defects by rendering the questions perfectly clear, and their subsequent answers as easy to be understood as the nature of the subject will admit.'

The first principle of the solution is to find the curve a minimum under a given area, instead of the area a maximum and the curve given, as in the question, which is the same thing. Mr. Emerson assumes another curve similar to the curve required, whose area is $\frac{2}{3}$, or $xy = 1$; and if $a = $ the parameter, then $ax = y^2$; from these two equations we have $x = \frac{1}{y}$, and $a = \frac{y^2}{x} = y^3$; also $\frac{y}{a} = \frac{y}{y^3} = \frac{1}{y^2} = xx$. Now the general expression for the length of the curve being $\sqrt{(\frac{1}{4}y^2 + \frac{y^4}{a^2})} + \frac{1}{4}a \times$ h. l. $\frac{2y}{a} + \sqrt{(1 + \frac{4yy}{ad})}$ by

writing y^3 for a in this expression, it becomes $\sqrt{(\frac{1}{4}y^4 + \frac{1}{y^2})} + \frac{y^3}{4}$

$\times$ h. l. $\frac{2}{y^4} + \sqrt{(1 + \frac{4}{y^4})} =$ a minimum; the fluxion of which being found in the common way, and made $= 0$, the equation gives z or

h. l. $\frac{2}{y^2} + \sqrt{(1 + \frac{4}{y^4})} = \frac{2}{yy} \sqrt{(1 + \frac{4}{y^4})}$; that is (putting x instead of $\frac{1}{y}$) $\frac{2xx}{3} \sqrt{(1 + 4x^4)} = $ h. l. $2x^2 + \sqrt{(1 + 4x^4)}$, which thrown into a series, and reverted, x is found; thence y, and the curve. Then, by sim. figures, he proportions thus, as the length of the curve thus found is to the length given in the question, so is x and y here found, to the abscissa and ordinate required.   H.

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Questions proposed in 1742, and answered in 1743.

1. QUESTION 226, by Mr. Tho. Ramsey.

There is a triangular field, near to two trees, an oak and a pine. One of the fences of the field, if produced 18 chains, would touch the oak; and another, if produced 12 chains, would touch the pine. The trees are 40 chains asunder, and a right line joining them is parallel to the third side of the field. Required the area when a maximum?

Answered by Hurlothrumbo.

Let AQ be parallel to CP; OP $= a = 40$; OA $= b = 18$; AQ $=$ PC $= c = 12$, and OQ $= x$. Then the area of the $\triangle$ AOQ is

$$\sqrt{\frac{2b^2c^2 + 2b^2x^2 + 2c^2x^2 - b^4 - c^4 - x^4}{16}};$$ therefore, as xx (OQ^2) :

$(a - x)^2$ (AC^2) :: the said area : the area of the $\triangle$ ABC, which, by the question, is a maximum; $\therefore$

$\frac{(a - x)^2}{x^4} \times$: $2b^2c^2 + 2b^2x^2 +$

$2c^2x^2 - b^4 - c^4 \cdot x^4$ is also a maximum: Whence $a(bb - cc)^2 - (bb + cc) ax^2 - (bb + cc) x^3 + x^4$
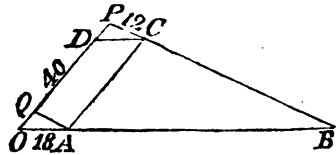
$= 0$: from which the value of x is found $= 7\cdot6986$: AC $= 32\cdot3014$; BC $= 50\cdot3473$; AB $= 75\cdot52$, and the area 61 a. 2 r. 9 p.

<center>Merones's <i>Answer.</i></center>

Draw DC parallel to AB; let PC $= b$, OA $= c$, PO $= a$, PD $= x$.

Then BC $= b\ \dfrac{a - x}{x}$, AB $= c\ \dfrac{a - x}{x}$, AC $= x\ \dfrac{a - x}{x} = a - x$;

and let $s = cc - bb$, $t = cc + bb$: Then, by the rule for finding

the area from the three sides, $\frac{1}{4}\sqrt{\dfrac{-ss + 2txx - x^4}{x^4}} \times (a - x)^2 =$

area a max. this in fluxions, and reduced, gives $x^5 - tx^3 - atx^2 + ass = 0$, and $x = 7\cdot698$, and the area $= 615\cdot57$.

This question is truly solved by the proposer Mr. *T. Ramsay*, Mr. *Ant. Thacker*, Mr. *Hemmingway*, Mr. *Farrer*, and *Tho. Cowper*, by a process much alike. But as I was at some trouble in solving it in a method different from any of them, before the receipt of theirs, I will, for variety's sake, here give it, though I don't think it at all better.

AQ $=$ PC $= 12$; AO $= 18$; OQ $= 2x$. The sum of the three sides is 30 $+ 2x$, half sum $= 15 + x$, the difference between half sum and three sides are $3 + x$, $x - 3$, and $15 - x$; then $\sqrt{((15+x)(15-x)(x+3)(x-3))}$, i. e. $\sqrt{(234x^2 - x^4 - 2025)}$ $= z =$ the area of OAQ. But as $xx : z :: (20 - x)^2 : z\ \dfrac{(20 - x)^2}{xx} =$ area of the $\triangle$ ABC $=$ a maximum; which in fluxions is $(\dot{z}(20 - x)^2 - 2\dot{x}z(20 - x))\ xx - 2\dot{x}z(20 - x)^2 = 0$; which divided by $x(20 - x)$, gives $(\dot{z}(20 - x) - 2\dot{x}z)x - 2\dot{x}z(20 - x) = 0$; or $20\dot{z}x - \dot{z}xx - 40\dot{x}z = 0$; $\therefore 20\dot{z}x - \dot{z}xx = 40\dot{x}z$; $\therefore \dot{z} = \dfrac{40\dot{x}z}{20x - xx}$;

but $234x^2 - x^4 - 2025 = z^2$; in fluxions, &c. $\dfrac{234\dot{x}x - 2\dot{x}x^3}{z} = \dot{z}$;

consequently $= \dfrac{40\dot{x}z}{20x - xx} = \dfrac{234\dot{x}x - 2\dot{x}x^3}{z}$; $\div$d by $2\dot{x}$ gives

$\dfrac{20z}{20x - xx} = \dfrac{117x - x^3}{z}$; $\therefore 20zz = (117x - x^3)(20x - x^2)$, or $20(234x^2 - x^4 - 2025) = (117x - x^3)(20x - x^2)$; in numbers is $4680x^2 - 20x^4 - 40500 = 2340x - 20x^4 - 117x^3 + x^5$; ordered, is $x^5 - 117x^3 - 2340x^2 + 40500 = 0$; from whence x is found $= 3\cdot8493$, and $2x = 7\cdot6986 = $ OQ.

<center>II. QUESTION 227, <i>by Mr.</i> Peter Kay.</center>

Supposing a pendulum, whose length is 29·2 inches, and its bob or

weight 27 pounds; to find at what part of the rod of that pendulum, a weight of one pound must be fixed, so as to have the greatest effect in accelerating the pendulum; or, so that the time of the vibration may be the shortest possible? The rod itself being supposed void of gravity.

Answered by Mr. Kay.

Put a = the length of the pendulum = 29·2 inches; w = weight of the bob 27 pounds; v = the weight to be fixed to the rod = 1 pound; and x = the required distance thereof from the point of suspension; then the distance of the centre of oscillation from that point will be $\dfrac{aaw + xxv}{aw + xv}$, which by the question ought to be a minimum; and therefore $2xxv \times (aw + xv) - vx (aaw + xxv) = 0$, whence x is $= a \dfrac{\sqrt{(xw + wv - w)}}{v} \doteq 14\cdot4686$ inches.

The same answered by Mr. J. Watts.

The momenta of all moving bodies are as a rectangle of their celerity and mass, and the celerity of a pendulum as its length, &c. Then the centre of oscillation of any compound pendulum (by Colin Mac Laurin's Fluxions, lib. 2, p. 453) is equal to the squares of the distances multiplied into their respective weights, and their aggregate divided by the sum of the momenta; therefore (the answer as above.)

III. QUESTION 228, by Mr. J. May.

The great inundations we have had here lately in Holland, has laid above six hundred thousand acres of land under water; and hath ruined and washed away the boundaries, that it is almost impossible again to determine each man's possessions; but to help a friend, and prevent disputes, your assistance is desired.

He had a piece of land, AFPVD, which was divided into two equal parts by the right line VF; of which the part ADVF was a geometrical square; and the other part VPF an apollonian parabola, V the vertex, and F its focus: but all was defaced except the side AD; which we with some difficulty measured fifty-two chains. So that to fix the boundaries again there is required the lengths of the sides VP and FP: It is also expected that the points V, F, P, &c. be determined by a geometrical construction.

Answered by Hurlothrumbo.

Upon the given line AD describe the square ADVF, and from the
focus F, and vertex V, the parabola
VP; bisect FV with the perpendi-
cular HG, and take HG $= \frac{1}{4}$AD, and
from the centre G with the radius
GV describe the circle VPF, and from
the point P, where it cuts the para-
bola, draw FP, then will ADVPFA
(by book 1, prop. 30, Newton's
Princip.) be the figure required.
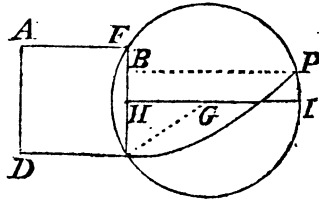
In this fig. a line drawn from F *to* P *is omitted, which the reader
may supply; the letter* V *is also wanting.*

Answered by Mr. Farrer.

Let $m =$ VF $= 52$ chains, $x =$ VB; then $m - x =$ BF, and per
conics, PB $= 2\sqrt{mx}$, and the area of VPF $= \dfrac{4x}{3}\sqrt{mx} + (m - x)$
$\sqrt{mx} = mm$; this equation reduced is $x^3 + 6mx^2 + 9m^2x - 9m^3$
$= 0$; hence $x = 34.77163$, BP $= 85.044$, FP $= 86.77157$, and the
arch VP $= 93.7442$ chains. And the points V, P, and F, will be de-
termined by the following construction.

Let HG $= \frac{1}{4}$ of VF bisect at right angles the side of the square,
and on the centre G, with GF describe the circle, which will cut the
curve in P.

Mr. J. May, the Proposer,

After an analytic answer, gives this geometrical construction:
Having $xx = 4ay$ from the property of the parabola, and $yx = 6aa$
$- 3ax$; then $yxx = 4ayy$, and $yxx = 6aax - 3axx$, whence $4ayy$
$= 6aax - 3axx$, divided by $4a$ is $yy = \frac{3}{2}ax - \frac{3}{4}xx$, for $\frac{3}{4}xx$ put its
value $3ay$, we have $yy = \frac{3}{2}ax - 3ay$, to which add $xx = 4ay$, is $yy +$
$xx = \frac{3}{2}ax + ay$ an equation to a circle, and $y = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}aa + \frac{3}{2}ax)}$;
put the surd $= 0$, then is $y = \frac{1}{2}a$; bisect FV in H, and draw HGI
perpendicular to VF, in which the centre of the circle will be; then
$x = \frac{1}{2}a \mp \sqrt{(1\frac{1}{6}aa)}$; make HG $= \frac{1}{4}a$; draw VG, this is $\sqrt{(1\frac{1}{6}aa)}$
and the centre is G; put $x = 0$, then $yy = ay + 0$, and $y = \frac{1}{2}a \mp \frac{1}{2}a$,
or $y = 0$, or $= a$, then V, F are two points in the periphery. On
the centre G with GV draw the circle which cuts the given parabola
in the required points.

IV. QUESTION 229, *by* Hurlothrumbo.

Supposing an homogenous fluid, equal in density and magnitude

to the earth, to revolve uniformly about an axis; so that the greatest diameter thereof may be just double the axis from pole to pole; to find the time of one entire revolution; with a general theorem for the solution of other questions of this nature?

Answered by the Proposer.

Let r be the time wherein a body would describe a circle about the earth, just above the surface, by means of its own gravity $=$ $1^h 24^m 45^s$, and let a be the arch of a circle whose radius is 1, and secant n, supposing the given ratio of the equatorial diameter to the axis be as n to 1. Then by Simpson's Mathematical Disserta-tions, we shall have $r \times \sqrt{\dfrac{(2n^2 - 2)\sqrt{(nn - 1)}}{(nn + 2)3a - 9\sqrt{(nn - 1)}}}$ for the exact time of one entire revolution; which therefore, when $n = 2$, or the equatorial diameter is just double the axis, will be $2^h 31^m 20^s$.

Merones's Answer.

Let $r =$ earth's radius, $f = 16\frac{1}{12}$ feet. That the equinoctial dia-meter may double the polar one, the centrifugal force must take away half the gravity; to do which any point in the equinoctial must in $1''$ describe the arch whose versed sine is $\frac{1}{2}f$; or which is the same thing, the arch itself will be $\sqrt{rf}$; and the periodic time 3.1416 $\sqrt{(4r \div f)} = 7147'' = 2^h$ fere; and therefore the body will revolve 12 times as fast as our earth. But to give an accurate solution to this problem, the decrease of gravity arising from the earth's figure, ought to be taken into consideration; but this would render the cal-culus very intricate, and too long for this place.

v. QUESTION 230, by Mr. J. Turner.

Suppose the bung diameter of a spheroidal cask were 40 inches, and its diagonal 48 inches; it is required to find the head diameter of the least spheroidal cask possible, having the abovesaid dimen-sions; and its content in ale gallons?

Answered by Hurlothrumbo.

Put the diagonal $= 48 = a$, half the bung diameter $= 20 = b$, and half the head diameter $= x$; then the content will be $\dfrac{2bb + xx}{1077} \times$ $8\sqrt{(aa - (b + x)^2)}$, ale gallons, which by the question is to be a minimum. Now by making the fluxion of $\dfrac{2bb + xx}{1077} 8\sqrt{(aa - (b+x)^2)}$,

or that of $(2bb + xx)^2 (aa — (b + x)^2) = 0$; we get $2a^2x — 2b^3$ — $4b^2x — 5bx^2 — 3x^3 = 0$; that is, in numbers — $16000 + 3008x$ — $100x^2 — 3x^3 = 0$; whence $x = 7·85$, or $x = 12·63$, or $x = — 53·8$, but none of these roots is the required value of x.

For let BDFH be a curve, whose abscissa AC is x, and ordinate $(2b^2 + x^2)^2 (aa — (b + x)^2)$, and it is evident that when the ordinate of this curve is a mini-mum, the cask will be so too; but it appears from what has been found above, that the or-dinate CD grows less and less, till x or AC be-comes 7·85 (because till then its fluxion, or — $16000 — 3008x — 100x^2 — 3x^3$, is a nega-tive quantity) after which it increases till x becomes $= 12·63$ (the fluxion being affirmative) and then decreases again continually, till x arrives at 20, its greatest value; in which circumstance it will be less than in any former position, as will easily appear upon trial; therefore the head diameter is equal to the bung diameter indefinitely near, and the required content 236·468 ale gallons.
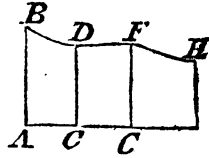
Mr. N. Farrer's *Answer.*

If $m = 40$, $n = 48$, $y = $ head diameter; then $\dfrac{m + y}{2}$ is the base of the right-angled triangle, and (per 47 Euc. I.) $\sqrt{\dfrac{4nn—mm—2my—yy}{4}}$ $= $ semi-length. Let $4nn — mm = d$; then its length is $\sqrt{(d — 2my — yy)}$, and its solidity $= (2mm + yy) \sqrt{(d — 2my — yy)} \times$ ·2618; whose fluxion $2\dot{y}y \sqrt{(d — 2my — yy)} + (2mm + yy)$ $\dfrac{— m\dot{y} — y\dot{y}}{\sqrt{(d — 2my — yy)}} = 0$. Reduced, $3y^3 + 5my^2 + 2mmy — 2dy$ $+ 2m^3 = 0$. Hence $y = 15·701465 = $ head diameter, $78·187894$ $= $ length; 250·174 ale gall. the content. And farther observes,

1. The cask will be greatest when it becomes a spheroid, (whose length is then 87·266 inches, and content 259·28 a. g.) and least when a cylinder (length 53·066, content 236·46 a. g.); therefore the question, properly speaking, does not admit of either maximum or minimum, for the least spheroidal cask will be infinitely near the cylinder, greatest near the spheroid. 2. If the length be 76, the content is 252·18 a. g. If 84 inches long, the content is 252·85, be-tween these two the least is that found above. 3. Between this least and the cylinder, there is another, whose capacity is a maximum.

Mr. Hemingway,

After his answer, remarks, That when any expression is put into

fluxions, the roots of the equation determine so many limits of increase and decrease of the flowing quantity; whence if there's but one root which answers the conditions of the problem, we obtain a maximum or minimum; but if there be more than one, the limits those roots exhibit may not determine either. In the present case, if the semi-diff. of diam. $= x$ increases gradually from its first value 7·3749, till it arrives at its second value 12·149, the cask continually decreases from 250·87 to 250·174; but x still increasing, the cask will increase again till it becomes a whole spheroid, whose content is 259·259 a. g. Again, let x gradually decrease from its first value, and the cask will decrease likewise, till it becomes a cylinder, whose length is 53·066 inches, and content, 236·47, which is the minimum, and may be considered as the middle frustum of a spheroid of an infinite length.

VI.   QUESTION 231, *by Mr.* N. Farrer.

On the 20th of December, 1740, at ten minutes past eight at night, I observed two noted fixed stars on the meridian; the difference of their altitudes thirty-seven degrees; and at ten o'clock I found their azimuths 31° 40′ and 61° 45′, both west, and their difference of altitudes 32°. Quere the latitude of the place of observation?

Solved.

This question happened to be over-limited, the latter given time of observation being superfluous, for it will not agree with the rest of the data; (the hour angle may be only 26° 30′, and the times of observation 14ᵐ after 8 and 10 o'clock).

Given
{
AB $= 37° =$ 1st diff. alt.
AZ $-$ BZ $= 32° =$ 2d diff. alt.
AZB $= 30° 5′ =$ diff. azim.
BZP $= 118° 15′ =$ zenith $\angle$.
ZPB $= 26° 30′ =$ hour $\angle$.
}

To find AZ $+$ BZ and PZ.

Put x and y for the sine and cosine of half the difference of AZ and BZ, s and q for those of half their sum. Then, by our new theorem, $sy + qx$ and $qy - sx$ is the sine and cosine of ZA, and $sy - qx$ and $qy + sx$ those of ZB; also let z represent the cosine of AZB, and d the cosine of AB. Then, by first theorem, $(qy - sx)(qy+sx)$ $+ (sy + qx)(sy - qx) z = d$; which when multiplied is $qqyy - ssxx + zssyy - zqqxx = d$, or $dqq + dss$; being transposed, $zssyy - ssxx - dss = dqq + zqqxx - qqyy$, $\therefore \dfrac{zyy - xx - d}{d + xxz - yy} = \dfrac{qq}{ss}$; to
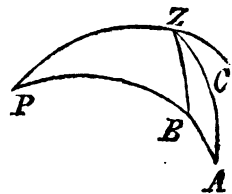
which add 1, it's $\dfrac{zyy - xx - d}{d + zxx - yy} + 1 = \dfrac{qq}{ss} + 1 = \dfrac{qq + ss}{ss}$, or

$\dfrac{z - 1}{d + zxx - yy} = \dfrac{1}{ss}$; $\because ss = \dfrac{d + zxx - yy}{z - 1}$, or $\dfrac{yy - d - zxx}{1 - z}$; $\therefore$

$2ss = \dfrac{2yy - 2d - 2zxx}{1 - z} = $ the versed sine of AZ + BZ.

<div align="center">Operation.</div>

	74°	16
	74	16
	148	32 Sum.

Whose ver. s. is $1\cdot8480481 = 2yy$	Whose vers. s. is $\cdot1519519 = 2xx$
Sub. $2d + 2xxz = 1\cdot7287543$	multip. by $z =$ $\cdot8652972$
$2yy - 2d - 2xxz = \cdot1192934$ rem.	gives $2xxz = $ $\cdot13148337$
this $\div$ d by $1 - z = \cdot1337028$ gives	add $2d =$ $1\cdot5972710$
the quotient $= \cdot8856064.$	Sum $2d + 2xxz = 1\cdot7287543$

which gives $= 83° \ 25' \ 53'' = $ AZ + BZ, the half of which $41° \ 42'$
$56''$ added to half the diff. 16°, gives $57° \ 42' \ 56'' = $ AZ the greater
side, and subtracted, leaves $25° \ 42' \ 56'' = $ BZ the lesser side; then
as s. $37°$: s. $30° \ 5'$ :: s. $57° \ 42' \ 56''$: s. $44° \ 44' = $ PBZ; again as
s. $26° \ 30'$: s. $25° \ 42' \ 56''$ :: s. $44° \ 44'$: s. $43° \ 13' = $ PZ the com-
plement of the latitude, $\therefore 36° \ 47'$ is the latitude required.

<div align="center">VII.　QUESTION 232, by Mr. Ant. Thacker.</div>

If $x^3 + y^3 - 945yx = 0$, express the nature of the curve AM, and
150400 be the area of the space AMP; it is required to find the area
of the greatest parallelogram that can be inscribed in the said figure
AMP ?

<div align="center">Answered by Hurlothrumbo.</div>

Put $a = 945$, $b = 150400$, PM $= y$, AP $= x = vy$; then, by
writing vy instead of x in the equation of the

curve, &c. we shall have $y = \dfrac{av}{1 + v^3}$, $x =$

$\dfrac{av^2}{1 + v^3}$, and $y\dot{x} = \dfrac{2a^2v^2\dot{v} - a^2v^5\dot{v}}{(1 + v^3)^3}$; whose fluent

is $\dfrac{a^2}{6} - \dfrac{a^2}{2(1 + v^3)^2} + \dfrac{a^2}{3(1 + v^3)}$, which, by

restoring x and y, will be $\dfrac{aa}{6} + \dfrac{xy}{2} - \dfrac{ayy}{6x}$ or
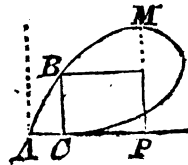
$\frac{xy}{2} + \frac{axx}{6y}$. Now by putting $z = 1 + v^3$, $\frac{aa}{6} - \frac{aa}{2(1 + v^3)^3} +$

$\frac{aa}{3(1 + v^3)} = b$, and $c = \frac{6b}{aa} - 1$, we shall get $z = \frac{1 - \sqrt{(1 - 3c)}}{c}$

$= 1\cdot512$; whence $v = 0\cdot8$, $y = 500 = $ PM, and $x = 400 = $ AP.
Hence the area of the greatest inscribed parallelogram PB, will be
found $= 92133.$

Note, This curve, having its equation affected alike by x and y
returns into itself, and has its convex and concave parts exactly si-
milar to each other.

VIII. QUESTION 233, *by Mr.* Rich. Piercy.

If $x^n \times (n - x)^x = y$; when n is $= 1000$; required the value
of x, y being the greatest number that possibly can be?

Answered by J. Watts.

Since $x^n (n - x)^x$ is to be a maximum, its h. l. $n \times$ l. $x + x \times$
l. $(n - x) = $ max. in fluxions $\frac{n\dot{x}}{x} + \dot{x} \times$ l. $(n - x) - \frac{x\dot{x}}{n - x} = 0$;

which divided by $\dot{x}$ is $\frac{n}{x} +$ l. $(n - x) - \frac{x}{n - x} = 0$, i. e. $\frac{1000}{x} +$

l. $(1000 - x) - \frac{x}{1000 - x} = 0$. By trials it is found x must be
greater than 800, $\therefore$ put $800 + z = x$, which substituted above,

gives $\frac{1000}{800 + z} +$ l. $(1000 - 800 - z) - \frac{800 + z}{1000 - 800 - z} = 0$;

contracted, it is $\frac{1000}{800 + z} +$ l. $(200 - z) - \frac{800 + z}{200 - z} = 0$; out of

fractions, is $200000 - 1000z + (160000 - 600z - zz) \times$ l. $(200 - z) - 640000 - 1600z - zz = 0$; or $- 440000 - 2600z - zz + (160000 - 600z - zz) \times$ l. $(200 - z) = 0$; signs changed, $440000 + 2600z + zz - (160000 - 600z - zz) \times$ l. $(200 - z) = 0$; ordered by infinite series, is $440000 + 2600z + zz - (160000 + 600z + zz) \times$ l. $200, - \frac{z}{200} - \frac{z^2}{80000} - \frac{z^3}{24000000} - \frac{z^4}{6400000000}$, &c. $= 0$; or finding the hyp. log. of 200, *viz.* $5\cdot2983$,
&c. its $440000 + 2600z + zz - (160000 - 600z - zz) \times : 5\cdot2983$

$$-\frac{z}{200} - \frac{z^2}{80000} - \frac{z^3}{24000000}, \text{ &c. } = 0; \text{ that is, } -407728 +$$

$6578\cdot98z + 5\cdot2983z^2 - \cdot005833z^3 - \cdot0000125z^4 = 0;$ or $6578\cdot98z$
$+ 5\cdot2983z^2 - \cdot005833z^3 - \cdot0000125z^4 = 407728;$ which divided
by the coefficient of z, (per Simpson's Flux. p. 101.) gives $z +$
$\cdot0008054z^2 - \cdot0000008z^3 - \cdot0000000001z^4 = 61\cdot9742,$ compared
with the series, in the same page, we have $b = \cdot0008054,$ $c =$
$\cdot0000008,$ $d = \cdot000000001,$ &c. and $z = 61\cdot9742$ which call y, then
will $z = y - by^2 + (2bb - c) \times y^3$ &c. $= 59\cdot37973$, which added
to 800, gives $x = 859\cdot37973.$

IX. QUESTION 234.

A father at his death bequeathed to his daughters these portions,
viz.

> To the eldest he gave $\cdot2\cdot^2$ of 1000 pounds.
> To the second $\cdot3\cdot^3$ of 1000 pounds.
> To the youngest $\cdot4\cdot^4$ of 1000 pounds.
> How much was each daughter's portion?

Answered by Mr. J. Watts.

Subtract the log. of $2=0\cdot3010300$ from the log. of $10 = 1\cdot0000000$,
the remainder is $\cdot69897000$, which multiplied by 2 is $\cdot1397940 =$
the log. of $\cdot2\cdot^2$, which subtract from unity, and the difference is
$\cdot8602060$, which sought in the logarithm tables, gives the absolute
number answering $\cdot72478$. And proceeding in like manner with $\cdot3\cdot^3$
and $\cdot4\cdot^4$, we shall have the eldest daughter's portion $724\cdot78 = 724l.$
16s. 4d. the second $= 696\cdot84 = 696l.$ 16s. 11d. and the youngest
$= 693\cdot14 = 693l.$ 2s. 8d.

THE PRIZE QUESTION, *by Mr.* Heath.

Admiral Vernon sailing on a south course, from Jamaica to Car-
thagena, sees Don Blass right before him, steering due west, along
the shore. He now continually bears directly upon him, in a right
line; when coming up with him, it appears that the Don had sailed
8 leagues during the chace, and that the said admiral was 7 leagues
distant from him when the chace began: Now, supposing each ship's
motion to be uniform during the whole chace, to find from thence
the distance sailed by Admiral Vernon?

Answered by Merones.

Let $cs = b = 8$, $AB = c = 7$; G, g be two points in the curve infinitely near each other, draw GF and gf parallel to AB, and to the points G, g draw the tangents GD, gd, and Do perpendicular to gd. Let $BD = x$, $BF = v$, and $GD = y$, $AG = z$, and $z = ax$, and consequently $\dot{z} = a\dot{x}$, where a is a constant, but unknown, quantity. By sim. triangles $\dot{z} : \dot{v} :: x$
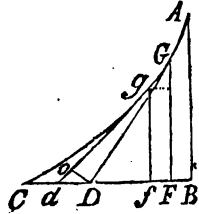
$\left(= \dfrac{\dot{z}}{a}\right) : do = \dfrac{\dot{v}}{a}$; then y is increased by

do, and decreased by Gg, that is $\dot{y} = \dfrac{\dot{v}}{a} - \dot{z}$; or $\dot{z} = -\dot{y} + \dfrac{\dot{v}}{a}$;

and the fluent is $z = c - y + \dfrac{v}{a} = ax$; and when they arrive at

c, $y = 0$, and $v = x = b$; whence $c + \dfrac{b}{a} = ab$; and $a = c +$

$\dfrac{\sqrt{(cc + 4bb)}}{2b} = 1\cdot529$; and therefore the whole curve $z = c + \dfrac{b}{a}$

$= 12\cdot2321$ leagues the distance sailed by Admiral Vernon.

Mr. Powle

Takes notice that this question is a particular case of prob. 8, p. 170, Simpson's Fluxions. And if $\dfrac{1}{x}$ be the ratio of the velocities of the two ships, and $AB = a$, and $BC = b$; then from the nature of the curve we have $\dfrac{ax}{1 - xx} = BC$, and $\dfrac{a}{1 - xx} = AC$ sought: Now

the equation $\dfrac{ax}{1 - xx} = b$ will give $x = \dfrac{\sqrt{(4bb + aa)}}{2b} - \dfrac{a}{2b}$, $\therefore$

$\dfrac{ax}{1 - xx} = \dfrac{2bb}{\sqrt{(4bb + aa - a)}} = 12\cdot2324 = AC$ required.

—●●●●●●—

Questions proposed in 1743, and answered in 1744.

I. QUESTION 235, *by Mr.* Thomas Cowper.

On the longest day, the distance from the branch of a tree to the

place where its shadow was projected upon the level ground was 97 inches; and $1^h 16\frac{1}{2}^m$ afterwards, the distance from another branch, $13\frac{1}{2}$ inches perpendicularly above the former, to the place where its shadow was projected upon the level ground was also 97 inches; and the length of the shadow at the time of the latter observation was $12\frac{2}{3}$ inches shorter than at the former: Required the latitude of the place and times of observation?

N. B. That 17 minutes must be deducted from the sun's altitude at the time of the first observation, and $16\frac{1}{2}$ at the second, as an allowance for his semi-diameter and refraction.

Answered by Mr. J. Turner.

There is given $AB = CD = b = 97$ inches, $AC = 13 \cdot 8 = c$, and $DB = 12 \cdot 6 = d$; to find $CG = x$. Put $y = BG$.

By 47 Euc. I. $\begin{cases} cc + 2cx + xx + yy = bb, \\ dd + 2dy + yy + xx = bb; \end{cases}$ consequently they are equal to one another, or $cc + 2cx = dd + 2dy$; and by the first equation, $y = \sqrt{(bb - cc - 2cx - xx)}$; which substituted for y, in $cc + 2cx - dd = 2dy$, is $cc + 2cx - dd = 2d\sqrt{(bb - cc - 2cx - xx)}$; and squaring both sides of the equation, makes $c^4 + 4c^3x - 2ccdd + 4ccxx - 4ddcx + d^4 = 4ddbb - 4adcc - 8ddcx - 4ddxx$; transposed, $4ccxx + 4ddxx + 4c^3x + 4ddcx = 4ddbb - 2ddcc - c^4 - d^4$; and dividing all by the coefficient of the highest power, it will stand, $xx + cx = \dfrac{4ddbb - 2ddcc - c^4 - d^4}{4cc + 4dd}$.

The square completed, and properly reduced, gives this theorem,

$x = \sqrt{\left(\dfrac{ddbb}{dd + cc} - \dfrac{dd}{4} \right)} - \dfrac{c}{2} = 58 \cdot 2 = CG.$ Thence is found $y = 65$, $GA = 72\frac{1}{4}$, and $DG = 77 \cdot 6$; whence by common trigonometry, may be found the angles $GAB = 42° 4' 30''$, $ABG = 47° 55' 30''$, and $CDG = 36° 52' 12''$. Now having the two altitudes of the sun found, and the difference of the times given per question $= 76\frac{1}{2}^m$, the azimuth is easily had; then by the theorem in the answer to question 220, the latitude will come out $52° 19' 37''$, and the times of the day $8^h 0^m 11^s$, and $9^h 16^m 23^s$ the answer required.

The same answered by Mr. Betts.

Put x and y for the sine and cosine of the angle EFD, radius $= 1$.

Then by trigonometry, as s. $\angle$EFD : ED ::
s. $\angle$EDF : EF, i. e. $x : d :: y : c$; therefore
$cx = dy$ and $x \div y = d \div c =$ tangent of
42° 23′ 51″ the angle EFD, or half the sum
of the angles BAD and BAF = half the sum of
the two altitudes of the sun. Again, sine
EDF : EF :: rad. : DF, i. e. $y : c :: 1 : c \div$
y; and as AD : radius :: $\frac{1}{2}$FD : sine of half
the angle DAF, viz. $b : 1 :: \dfrac{c}{2y} : \dfrac{c}{2by}$; or
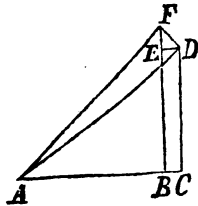
as $2b : c :: \dfrac{1}{y} : \dfrac{c}{2by}$, viz. $2b : c ::$ secant 42° 23′ 51″ : sine 5°
31′ 39″ equal to half the angle DAF, which is half the difference of
the required altitudes; whence the altitude CAF = 47° 55′ 30″, and
CAD = 36° 52′ 12″, and the times of the day 8^h and 9^h 16^m, the la-
titude 52° 19′ 37″, the answer required.

Additional Solution.

Make DB = the given diff. of the shadow [Fig. 1.], and perpendicu-
lar thereto DE = the given diff. of the heights of the boughs; then
from E and B as centres, with the same given distance from each
bough as a radius, describe two circular arcs, and having drawn
EA, and BA to the point of intersection, A, draw DC $\parallel$ and = EA,
and AC drawn to meet DB, produced, in G, will determine CG and
AG for the two required heights of the boughs.——DC being, by con-
struction, = EA = BA, AC (by Euc. I. 34.) = and $\parallel$ DE and conse-
quently the angle at G = the angle at D = a right one.   H.

II.   QUESTION 236, by Hurlothrumbo.

If two bodies, L and T, whose masses are respectively equal to
those of the moon and earth, were projected at the same time, and
in the same plane, from two places, A and B, at the distance of a
hundred thousand miles from each other; the former, L, with a ve-
locity of 5 miles per second, making an angle with AB of 100 degrees,
and the latter with a velocity of 2 miles per second, an angle (on
the same side AB) of 60 degrees; it is required to find the distance
and position of the two bodies with respect to each other, also with
respect to the points A and B, after they have been 48 hours in mo-
tion; supposing them, when in motion, to be only acted upon by
each other.

Answered.

After the two bodies have been 48 hours in motion, T is distant

from the point of projection (B) 393270 miles, and makes an angle with it of 59° 55′. Also L makes with A an angle of 98° 23′, and is distant from it 861570 miles. Through the industrious labours of some of our correspondents we have got an answer to this abstruse and curious philosophical problem; but as it is doubted whether some errors are not in it, and believing no one so equal to the task as the author, whose knowledge and penetrability in such difficult and uncommon problems, is scarce to be rivalled in an age; we have given only the numbers above (besides the scheme and answer we have inserted in our first vol. of Diary Questions, p. 193.)* till such time as the proposer favours us with his.

Solution taken from the Mathematician.

Let L and T represent the bodies as projected, whose masses are respectively equal to those of the moon and earth, or in the ratio of unity to 39·778; and let AV and BV, the absolute directions of the two bodies meet each other in V: then, in the triangle ABV are given all the angles and the side AB (= 10000 miles) whence AV will be found equal to 253208·8163, and BV = 287938·4609 miles respectively.
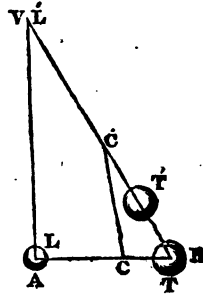
Now as 5 miles, the absolute space described by the body L in one second, is to 2 miles, the space described by the body T in the same time, so is the measure of AV, in miles, to 101283·5265, the measure of the space in miles, which T would uniformly describe in the time that L uniformly describes the space AV.

Let this space be denoted by BT′, then it is manifest, when the body L, by an uniform motion hath described the space AV, and is arrived at V, the body T will have described the space BT′ in the same time, and arrived at T′; therefore if T′V (= BV — BT′ = 287938·4609 — 101283·5265 = 186654·9344) be divided at c′ in the given ratio of their masses inversely, or c′T′ be taken to T′V in the ratio of unity to 1 + 39·778, then c′ will be the common centre of gravity (of the two bodies) in this position, and the distance c′T′ will be found = 4577·344; which added to BT′ gives BC′ = 105860·8705, whence VC′ = 18077·5904.

Moreover, let BC be taken to BA, in the ratio of unity to 1 + 39·778, and then C will be the point of equilibrio, or centre of gravity of the two bodies L and T, at the instant of projection, and consequently the space CB will be found = 2452·3027 and AC = 97547·6973. Now, if the right line CC′ be drawn, it is plain that in the time the bodies L and T would respectively describe the spaces

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* The first vol. of questions here referred to, is Thacker's Miscel. where the calculation of this question is given. N.

AV and BT′, their common centre of gravity will describe the space cc′; therefore in the triangle Bcc′, there being given the two sides BC, BC′, and the included angle CBC′ $= 60°$, the angle Bcc′, shewing the direction of the path of the centre of gravity, will be found $= 118°$ 52′, as also the space cc′ (104656·2697 miles) described by the same centre in the time that L uniformly describes the space AV. Hence, it will be as AV : cc′ :: 5 miles the space described by L in one second, to 2·0666 miles the space described by the centre of gravity in the same time.

Furthermore, draw BF equal and parallel to cc′ and let the parallelogram be completed, and join V, F′ and F, F′; then in the triangle F′c′v are given the two sides c′F′ (AC), c′V and the include angle F′c′v (60°) whence the angle c′F′v the relative direction of L with regard to the common centre of gravity, will be found 86° 44′ 50″ and the side F′v $=$ 157938·2468 miles the relative distances; hence it will be AV : VF′ :: 5 miles the absolute velocity of the body L in one second : 3·118735 miles, the relative velocity in the same time.
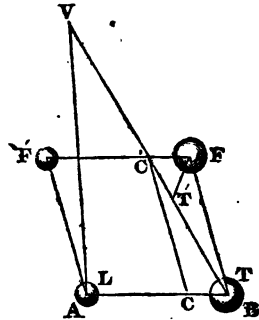
Moreover, since the triangles Vc′F′, and Fc′T′ have one angle c′ common, and the sides about that angle proportional, those triangles are similar, therefore the bodies L and T will describe similar figures (which in this case are hyperbolas) about their common centre of gravity; and the space c′F being to c′F′ as 1 : 39·778; the relative velocity of T will be that of L in the same ratio.

Let now F′LR represent the trajectory, or conic section described by the body L, to which F′v will be a tangent at the point F′; and let F′c′L be the area described about c′ the focus, or centre of force, in 48 hours the proposed time, and let cc′ ($=$AF′ $=$ BF) be the distance which the common centre of gravity, or the plane where the motions are performed, is carried uniformly in that time; also let Q be the vertex of the trajectory and draw c′Q, to which draw F′P and LN perpendicular, putting the sine of 86° 44′ 50″, the angle c′F′v (to the radius 1) ($=$·9983897 $= s$; 3982 the number of miles in the earth's radius $=b$; 0·003046, the parts of a mile which a heavy body will descend in a second of time at the earth's surface $= r$; $\sqrt{r} = 0·0051913$; 97547·6973 (c′F′) the distance of the point of projection from the centre of force $= d$; 3·118735, the relative velocity of L in a second, $= v$; the trans-

verse diameter of the described section $= 2a$; conjugate $= 2c$; ec-centricity $= e$; then (by pa. 26, Simpson's Essays) we have $2a = d$
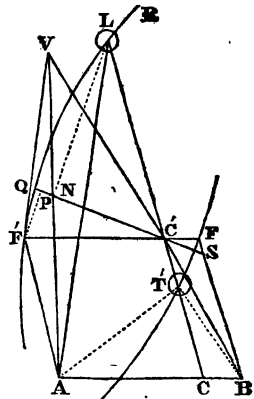
$\div \left(\dfrac{dv^2}{4 r b^2} - 1 \right) = 24941 \cdot 05$; $2c = \dfrac{\sqrt{(2a)}\, vsd}{b\sqrt{r}} = 218263 \cdot 41$; $e =$

$\sqrt{(a^2 + c^2)} = 109841 \cdot 9$; whence the distance of the focus from the vertex $= e - a = 97371 \cdot 38 \; (= g)$.

Now to find the area described about the centre of force in 48 hours, put $QN = x$, $LN = y$; the names of the given lines remaining as above: then, because by the property of the curve $\dfrac{c^2 \times (2ax + x^2)}{a^2}$

$= y^2$, we have $x = \dfrac{a\sqrt{(c^2 + y^2)}}{c} - a$; therefore $\dot{x} = \dfrac{ay\dot{y}}{c\sqrt{(c^2 + y^2)}}$,

and $y\dot{x} = \dfrac{ay^2\dot{y}}{c\sqrt{(c^2 + y^2)}} = \dfrac{a}{c} \times \dfrac{y^2\dot{y}}{\sqrt{(c^2y^2 + y^4)}} = \dfrac{a}{c}$ into

$\dfrac{\frac{1}{2}c^2 y\dot{y} + y^3\dot{y}}{\sqrt{(c^2y^2 + y^4)}} - \dfrac{\frac{1}{2}c^2 y\dot{y}}{\sqrt{(c^2y^2 + y^4)}}$, the fluxion of the area QLN; whose

fluent $\dfrac{a}{c}$ into $\dfrac{\sqrt{(c^2y^2 + y^4)}}{2} - \dfrac{c}{2} \times$ hyp. log. $\dfrac{y + \sqrt{(c^2 + y^2)}}{c} =$

the area of QLN, to which adding $\dfrac{(g-x) \times y}{2}$, the area of the triangle

CLN, we shall have $\dfrac{(g + a)y}{2} - \dfrac{ac}{2} \times$ hyp. log. $\dfrac{y + \sqrt{(c^2 + y^2)}}{c}$

for the area of the hyperbolical sector QCL. But since $F'c'$ is given $= d$ by the property of the curve, it will be as $e : a :: a + d - e : QP$ $:: d - g : QP \; (= 40 \cdot 0351)$ because $a - c = - g$, as demonstrated in de l' Hospital's Conic Sections, prop. 1, book 3; whence $F'P$ will be found $= 6493 \cdot 1924$; which let be denoted by m, and substituted instead of y in the general expression for the area of the hyperbolical

sector, and there will come out $\dfrac{(g + a)m}{2} - \dfrac{ac}{2} \times$ hyp. log.

$\dfrac{m + \sqrt{(c^2 + m^2)}}{c}$ for the area of the sector $QC'F'$, which added to

that of the former gives $\dfrac{(g + a)\,(m + y)}{2} - \dfrac{ac}{2} \times$ hyp. log.

$\dfrac{(m + \sqrt{(c^2 + m^2)}) \times (y + \sqrt{(c^2 + y^2)})}{c^2}$ the area described about

the centre of force in 48 hours, the given time; which area is like-wise found by multiplying $\dfrac{vsd}{2}$, the area described in one second,

by R the number of seconds in the given time, wherefore $\dfrac{(g+a)(m+y)}{2}$

$- \dfrac{ac}{2} \times$ hyp. log. $\dfrac{(m + \sqrt{(c^2 + m^2)}) \times (y + \sqrt{(c^2 + y^2)})}{2} =$

$\dfrac{v s R d}{2}$, from whence y will be found $= 499651\cdot2$ and consequently the distance (LC′) of the body L from C′ the centre of force $= 502296$: but LC′ : T′C′ :: $39\cdot778$: 1, therefore $\text{T}'\text{C}' = \dfrac{\text{LC}'}{39\cdot778} = 12627$, and consequently T′C′ $+$ LC′ the absolute distance of the bodies T′, L from each-other $= 514923$.

Since in the two right-angled triangles C′LN, C′F′P there are given two sides in each of them, the angles LC′N and NC′F′ will be found $=$ $84°\ 7'$ and $3°\ 49'$ respectively; the sum of both which is $87°\ 56'$ the true anomaly of each body, or the angle described about the centre C′ in the given time; but the distance C′C uniformly described by the centre of gravity of the two bodies in that time is $2\cdot0666 \times 48$ $\times\ 60 \times 60 = 35710\cdot848$ miles; therefore if AC′ and BT′ be drawn, in the triangle AC′F′ will be given the two sides AF′ ($= $ CC′), F′C′ and the angle AF′C′ ($= 180° - 118°\ 52'$) $61°\ 8'$; whence AC $= 103020$, the angle C′AF′ $= 56°\ 2'$, and consequently the angle AC′F′ $= 62°$ $50'$; therefore in the triangle AC′L are given the two sides AC′, C′L and the included angle AC′L $= 150°\ 46'$, whence the distance of the body L from the place of projection A will be found $= 594320$ and the angle C′AL $= 24°\ 23'$ and consequently the angle BAL, or the position of the body L $= 87°\ 13'$: Lastly, in the triangle AC′T′ are given the two sides AC′, C′T′ and the included angle AC′T′, whence AT′ $= 97489$ and the angle T′AC′ $= 6°\ 10'$; therefore in the triangle ABT are given the two sides AB, AT′ and the contained angle BAT′, whence BT′ the distance of the body T′ from the place of projection B will be found $=$ 93785 and angle of position ABT′ $= 60°\ 17'$.

III. QUESTION 237, *by Mr.* Nich. Oats.

A fleet of ships, at Portsmouth, is bound with military stores, &c. for our brave admiral in the West Indies ; and being informed, by experienced navigators, that a ship, in sailing upon a wind, having her larboard tacks on board, which makes her way good six miles an hour, will, when got into a trade wind (which blows in the latitude of thirty) make her way nine miles an hour; now admit the fleet can sail at the rate above, I demand the course and distance, before and after their arrival in the trade wind, to be performed in the shortest time possible, from the Lizard to Jamaica, and the minimum, according to Wright's projection. Lizard in lat. $49°\ 56'$, long. $5°\ 14'$ west, Jamaica $18°$ and $76°$?

Answered by Bironnos.

Let LN represent the enlarged diff. lat. between the Lizard and

Jamaica, LS that between the Lizard and the trade wind. Then we have LS $= a = 1579\cdot9$; TK $= 790\cdot2 = b$; T$b = 720 = q$; JN $= 4246 = n$. Let $y =$ TS; then $n - y =$ JK; and per
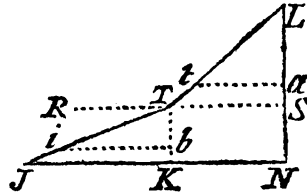
sim. triang. $a : y :: m : \dfrac{my}{a} = ta$;

$b : n - y :: q : q\dfrac{n - y}{b} = ib$; and

per 47 Euc. I. $\sqrt{\left(m^2 + \dfrac{m^2 y^2}{a^2}\right)} =$ LT; also $\sqrt{\left(qq + qq\dfrac{(n - y)^2}{b^2}\right)}$

$=$ TI; hence the required minimum is $\dfrac{m}{6a}\sqrt{(aa + yy)} + \dfrac{q}{9b} \times$

$\sqrt{(bb + (n + y)^2)}$; which being put into fluxions, there comes out $(qn - qy)2a\sqrt{(aa + yy)} = mq \times 3b\sqrt{(bb + (n - y)^2)}$; which solved, $y = 1857\cdot9215 =$ TS. Hence the course from the Lizard to the trade wind is south $49^\circ 37' 25''$ westerly, distance $= 1846\cdot23$ miles; and thence to Jamaica S. $71^\circ 41' 27''$ W, distance $2291\cdot95$ miles.

Answered by Mr. J. Powle.

RS is the parallel of 30°, L the Lizard, J Jamaica, T the point where the fleet must come to in their way, the $\angle$ TLS the first course steered, &c. (per Wright's projection) LS $= 1597\cdot8979$ miles, which call b; SN $=$ TK $= 790\cdot1607 = c$; JN $= 4246 = d$. Put $x =$ TS; $m = 6$; $n = 9$: then (per 47 Euc. I.) TL $= \sqrt{(bb + xx)}$; TJ $= \sqrt{(cc + dd - 2dx + xx)}$; and by a uniform velocity, the times of description will be $\dfrac{\sqrt{(bb + xx)}}{m}$, and $\dfrac{\sqrt{(cc + dd - 2dx + xx)}}{n}$, the sum of which two must be a minimum; the fluxion made $= 0$, and ordered, there will arise this equation:

$$\begin{matrix} & & & + ddnn & & & \\ + n^2 & & - 2dn^2 & + ccnn & & & \\ & x^4 & x^3 & & xx + 2dbbm^2\,x = bbddm^2 \\ - m^2 & & + 2dm^2 & - bbm^2 & & & \\ & & & - ddm^2 & & & \end{matrix}$$

In numbers, $52x^4 - 441584x^3 + 771997102\cdot9985x^2 + 763080806290\cdot03x = 1620020551753144$.

Solved, $x = 1458\cdot016$, or $2891\cdot56$, or $5488\cdot45$, or $1346\cdot03$; but it is the first of these values ($1458\cdot016$) which serves our present purpose. Now having found TS, by plain trigonometry TL $= 2149\cdot86$ is had (being the distance sailed before the fleet's arrival in the trade wind) and the course steered S.W. by S. $8^\circ 27'$ westerly: Thence to Jamaica TJ is $2897\cdot79$; course W.S.W. $6^\circ 40'$ westerly.

IV. QUESTION 238, *by Mr.* Peter Kay.

To find in what arc of a circle a pendulum must vibrate, so that the time of one whole descent shall be equal to the time in which a heavy body would fall along the chord of the same arc.

Answered by Merones.

Put c = chord of the arch required; x = any variable part of it; r = the length of the pendulum; t = time of the descent in the chord; and z = time of the descent in the arch. Then (by ex. 10, prop. 13, Mr. Emerson's Doctrine of Flux. p. 114.) $\dot{z} = \dfrac{tr}{2} \times$

$$\frac{\dot{x}}{\sqrt{(cx - xx)}\,\sqrt{(4rr - cx)}} = \frac{t\dot{x}}{4\sqrt{(cx - xx)}} \times : 1 + \frac{cx}{2.4r^2} + \frac{3.c^2x^2}{2,4.16r^4}$$

$+ \dfrac{3.5.c^3x^3}{2.4.6.64r^6}$ &c. And by forms 10 and 17 (pages 62, 68, ibid.)

of the table, $z = \dfrac{tt}{2} \times$ arch whose sine is $\sqrt{\dfrac{x}{c}} \times : 1 + \dfrac{cc}{16r^2} +$

$\dfrac{3^2c^4}{4.16^2r^4}$ &c. $=$ (when $x = c$) $\dfrac{3.1416t}{4} \times : 1 + \dfrac{cc}{16r^2} + \dfrac{3^2c^4}{2^2 16^2 r^4}$ &c.

$= t$, per quest. $\therefore$ (putting $r = 1$) $cc + \dfrac{3^2c^4}{2^2 16} + \dfrac{3^2 5^2 c^6}{2^3 3^2 16^2} + \dfrac{3^2 5^2 7^2 c^8}{2^2 3^2 4^2 16^3}$

&c. $= 4.3744$: whence, by reversion, $cc = 2.5107$; and $c = 1.5846$, the chord of $104° 48'$ the arch through which the pendulum must descend.

The same answered by the Proposer, Mr. Peter Kay.

Let $p = 3.14159$; a = length of the pendulum; c = the versed sine of the arch described in the descent; then the time of descent, or half the time of vibration, will be $\dfrac{a^{\frac{1}{2}}p}{\sqrt{2}} \times : 1 + \dfrac{c}{2.2.2a} +$

$\dfrac{c^2}{2.2.4.4.4a^2}$ &c. and the time along the chord by $2\sqrt{2a}$; (as demon. p. 140, 141, of Simpson's Flux.) which two expressions must by the question be equal to each other; divide both by $\dfrac{a^{\frac{1}{2}}p}{\sqrt{2}}$, and put $x = \dfrac{c}{2a}$ and the equation will become $1 + \dfrac{x}{2.2} + \dfrac{3.3x^2}{2.2.4.4}$ &c. $= \dfrac{4}{p}$,

and therefore $x + \dfrac{3.3x^2}{4.4} + \dfrac{3.3.5.5x^3}{4.4.6.6} + \dfrac{3.3.5.5.7.7x^4}{4.4.6.6.8.8}$ &c. $\left(\dfrac{16}{p} - 4\right)$

$= 1\cdot093$; whence $x = c \div 2a$, will be found $= \cdot642$, and therefore $c \div a = 1\cdot284$; and consequently the required arch $= 106° \, 32'$.

<div style="text-align:center">

v.   QUESTION 239, *by Mr.* Robert Heath.

</div>

> Four maids, wise and fair, and as Astrea rare, -
>   (Impatient to tie Hymen's noose)
> Burn with amorous flame for an artist of fame:
>   Then which for her age would you choose?

Supposing their several ages to be represented by the following equations.

$$a \times : ee + uu + yy = 20850.$$
$$e \times : aa + uu + yy = 23238.$$
$$u \times : aa + ee + yy = 24654.$$
$$y \times : aa + ee + uu = 24750.$$

What was each of their ages, and the analytical investigation?

<div style="text-align:center">

Solution.

</div>

Put $ma = e$, $na = u$, $ra = y$, $b = 20850$, $c = 23238$, $d = 24654$, $e = 24750$; then

$$a^3(m^2 + n^2 + r^2) = b \qquad ma^3(1 + n^2 + r^2) = c$$
$$na^3(1 + m^2 + r^2) = d \qquad ra^3(1 + m^2 + n^2) = e.$$

Whence $a^3 = \dfrac{b}{m^2 + n^2 + r^2} = \dfrac{c}{m(1 + n^2 + r^2)} = \dfrac{d}{n(1 + m^2 + r^2)}$

$= \dfrac{e}{r(1 + m^2 + n^2)}$, and consequently

$$mb(1 + n^2 + r^2) = c(m^2 + n^2 + r^2)\ldots\ldots\ldots\ldots(1)$$
$$nb(1 + m^2 + r^2) = d(m^2 + n^2 + r^2)\ldots\ldots\ldots\ldots(2)$$
$$rb(1 + m^2 + n^2) = e(m^2 + n^2 + r^2)\ldots\ldots\ldots\ldots(3)$$

From equation (1) $n^2 + r^2 = \dfrac{cm^2 - mb}{mb - c}$, and if m^2 be added to both sides we have $m^2 + n^2 + r^2 = \dfrac{m^3 b - mb}{mb - c}$. Put this $= s$, then the second and third equations become

$nb(1 + n^2 + r^2) = ds$, and $rb(1 + m^2 + n^2) = es$; from whence

$$1 + n^2 + r^2 = \dfrac{ds}{nb} \text{ and } 1 + m^2 + n^2 = \dfrac{es}{rb}.$$

But since $m^2 + n^2 + r^2 = s$, $1 + m^2 + r^2$ is $= s + 1 - n^2$ and $1 + m^2 + n^2 = s + 1 - r^2$;

therefore $s + 1 - n^2 = \dfrac{ds}{nb}$, and $s + 1 - r^2 = \dfrac{es}{rb}$.

Taking the sum of the squares of these two equations, and also their product, we have

$$2(s + 1)^2 - 2(s + 1)(n^2 + r^2) + n^4 + r^4 = \frac{(d^2 + e^2)s^2}{(n^2 + r^2)b^2}$$

and $(s + 1)^2 - (s + 1)(n^2 + r^2) + n^2r^2 = \dfrac{des^2}{nrb^2}$.

But $n^4 + r^4 = (n^2 + r^2)^2 - 2n^2r^2$, and $n^2 + r^2 = s - m^2$; therefore the above equations become

$$2(s+1)^2 - 2(s+1)(s - m^2) + (s - m^2)^2 - 2n^2r^2 = \frac{(d^2 + e^2)s^2}{(s^2 - m^2)b^2}$$

and $(s + 1)^2 - (s + 1)(s - m^2) + n^2r^2 = \dfrac{des^2}{nrb^2}$.

Now to exterminate n and r from these equations, let twice the second equation be added to the first, and there results

$$(2 + s - m^2)^2 = \left(\frac{d^2 + e^2}{s - m^2} + \frac{de}{nr} \right) \frac{s^2}{b^2},$$

therefore $\dfrac{b^2(2 + s - m^2)^2 (s - m^2) - (d^2 + e^2)s^2}{(s - m^2)b^2} = \dfrac{des^2}{nrb^2}$

and $\dfrac{de(s - m^2)}{b^2(2 + s - m^2)^2 (s - m^2) - (d^2 + e^2)s^2} = nr$.

Again, let twice the second be subtracted from the first, and we have

$$(s - m^2)^2 - 4n^2r^2 = \frac{(d^2 + e^2)s^2}{(s - m^2)b^2} - \frac{des^2}{rnb^2}$$

or $\dfrac{b^2(s - m^2)^3 - (d^2 + e^2)s^2}{(s - m^2)b^2} = 4n^2r^2 - \dfrac{des^2}{rnb^2}$,

and substituting for nr and $\dfrac{des^2}{rnb^2}$ their values found above, we have, after proper reduction,

$$d^2e^2(s - m^2)^3 = b^2(s + 1 - m^2) \left\{ b^2(2 + s - m^2)^2 (s - m^2) - (d^2 + e^2)s^2 \right\}^2$$

in which if we substitute for s its value $\dfrac{m^2b - mb}{mb - c}$ we have

$$md^2e^2(mb-c)^7 = b^6(cm^2 + c) \left\{ (cm^2 + bm + c)^2 (cm-b) - d^2e^2(m^2 - 1)(m^2b - mc) \right\}^2$$

an equation containing only m and known quantities, and which being expanded produces an equation of the 12th power from whence

m is found $= \frac{6}{5}$; then $s = \frac{1390}{225}$, and $s - m^2 = \frac{1066}{225}$; therefore n^2

$+ r^s = \frac{1066}{225}$ and $rn = \frac{7}{3}$, consequently n is found $= \frac{7}{5}$ and $r = \frac{5}{3}$;

therefore $a = \sqrt[3]{\frac{b}{3}} = 15$, $e = ma = 18$, $u = na = 21$, and $y = ra = 25$. L.

The solution of this question, as given by Mr. Ash, is omitted, us being so very erroneous.

VI. QUESTION 240, *by* F. R. S.

Let there be the frustum of a cone, whose less diameter is 20 inches, its greater 40, and length 90; which being cut by a plane diagonally through the contrary extremities of its two diameters, will divide it into two parts (called hoofs) a greater and a less. There is required a scientific theorem for finding the solid content of each part; there being none yet given by any author (except perhaps those got from a tedious series) which will give the contents precisely true, when at the same time we have found by an analytical method, the true solidity of each. All which may be made fully appear, and shall be demonstrated in the next year's Diary.

Answered by F. R. S.

In the given frustum of the cone let $AB = D = 40$; $gG = d = 20$; $GP = h = 90$. And put $AG = t$ the transverse diameter; $nm = c$ the conjugate; $GB = z$; and $cc = x$. Extend the transverse to c, and draw the two perpendiculars to it cc and iB, also $f = GC$, and $x = cc$. Then by similar triangles $t : h :: D :$

$$\frac{Dh}{t} = Bi, z : \frac{Dh}{t} :: f : \frac{Dhf}{zt} = cc, \frac{D-d}{2}$$

$$: z :: \frac{d}{2} : \frac{dz}{D-d} = f. \quad \text{Substitute} \quad \frac{dz}{D-d}$$

for f, then $cc\ (x) = \frac{Ddh}{Dt - dt}$, and x is nearly 38; which multiplied by $\frac{1}{3}$ of the area of the ellipsis, $AnGm$, viz. $\frac{\cdot7854 tc}{3}$ or

ntc, gives $\frac{Dhdnc}{D-d}$, the solidity of $ACGmA$;
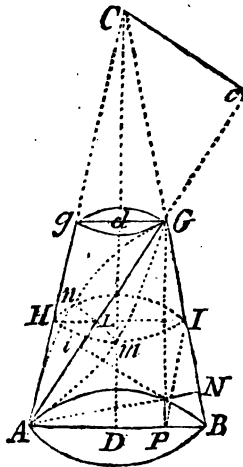
from which take the solid gCG, i. e. $\dfrac{ddnhd}{\text{D}-d'}$ leaves $\dfrac{\text{D}hdnc-d^3nh}{\text{D}-d}$ the content of AgGA. But $nm\ (=c)$ is known to be a mean proportional between AB and gG; i. e. $\sqrt{\text{D}d}=c$; whence $\dfrac{\text{D}hdn\sqrt{\text{D}d}-d^3nh}{\text{D}-d}=$ lesser hoof AgGA. Also $\dfrac{\text{D}^3hn-\text{D}dn\sqrt{\text{D}d}}{\text{D}-d}=$ greater hoof ABGA.

But in this example, as $2d=\text{D}$, the theorems are reducible, $\dfrac{\text{D}\sqrt{\text{D}d}-dd}{\text{D}-d}\times \cdot 2618dh$ is $\dfrac{2d\sqrt{2dd}-dd}{2d-d}\times \cdot 2618dh$, i. e. $(2dd\sqrt{2}-dd)\times \cdot 2618h$, or $(2\sqrt{2}-1)\times \cdot 2618hdd$. In numbers, $1\cdot 8284\times 90\times 400\times \cdot 2618=17232\cdot 304$ the lesser hoof.

And $\dfrac{\text{D}^2-d\sqrt{\text{D}d}}{\text{D}-d}\times \cdot 2618\text{D}h$ becomes $\dfrac{4dd-dd\sqrt{2}}{2d-d}\times \cdot 2618\times 2hd$, that is, $(4dd-dd\sqrt{2})\times \cdot 2618\times 2h$, or $(4-\sqrt{2})\times \cdot 2618\times 2hdd$: but $\sqrt{2}=1\cdot 4142$, and $4-1\cdot 4142=2\cdot 5858$; therefore in numbers $2\cdot 5858\times 2618\times 180\times 400=48741\cdot 2956$ the greater ungula. Which ungulas together make the whole frustum $65973\cdot 6$.

The same answered by Merones.

Let L be the centre of the ellipsis AnGm, mLn the common section of it and the circle HnIm, draw Ip parallel to HA, &c. Since GA is bisected in L, $\therefore$ LI $=\frac{1}{2}$AB, and LH $=\frac{1}{2}g$G; whence nL$^2=$ HLI $=\frac{1}{4}g$G $\times$ AB, and $nm=\sqrt{(g\text{G}\times \text{AB})}$; and the area of the ellipsis $c\times$ GA$\sqrt{(g\text{G}\times \text{AB})}$ (c being $=\cdot 7854$). By similar triangles, AB $-g$G $:$ G$p::$ AB $:$ CD $=\dfrac{\text{AB}\times \text{G}p}{\text{AB}-g\text{G}}::g$G $:$ $cd=\dfrac{g\text{G}\times \text{G}p}{\text{AB}-g\text{G}}$; and, by the same, GA $:$ gG $::$ CD $:$ $cc=\dfrac{\text{CD}\times g\text{G}}{\text{GA}}$; $\therefore$ the area ellipsis $\times$ $cc=\dfrac{c\times \text{CD}}{3}\sqrt{(g\text{G}^3\times \text{AB})}=$ solidity of GCA; but $c\times g$G$g\times cd=$ solidity Gcg; and $\dfrac{c\times \text{AB}^2\times \text{CD}}{3}=$ solid BCA; whence $\dfrac{c\times \text{CD}}{3}\times$ $(\text{AB}^2-g\text{G}\sqrt{(g\text{G}\times \text{AB})})=$ ungula GAB; and $\dfrac{c\times \text{CD}}{3}\times \sqrt{(g\text{G}^3\times \text{AB})}-\dfrac{c\times \text{CD}}{3}gGg=$ ungula GgA. Hence, putting the height G$p=$ H, AB $=$ B, gG $= b$, $c=\cdot 7854$; then $\dfrac{c\text{HB}}{\text{B}-b}\times \dfrac{\text{B}^2-b\sqrt{\text{B}b}}{3}=$ the

greater ungula $\text{GAB} = 48740\cdot55$; and $\dfrac{\text{CHB}}{\text{B}-b} \times \dfrac{b\sqrt{\text{B}b}-bb}{3} = $ the lesser ungula $\text{GGA} = 17232\cdot055$.

Bironnos has answered this Question in a method something different from the two last.

Let fall the $\perp$ AN upon GB; put $2a =$ greater diameter AB, $2q =$ lesser $\text{G}g$. Their half sum $\text{AP} = m$; half diff. $\text{PB} = n$; the frustum's height $\text{GP} = d$; and $\cdot785398$, &c. $= s$; then $\sqrt{(d^2 + m^2)} = $ AG the transverse, and $2\sqrt{aq} = $ the conjugate: Then, per sim. $\triangle$s, PB

$:$ GP $::$ gd $:$ $cd = \dfrac{dq}{\text{g}}$, and $::$ BD $:$ DC $= \dfrac{da}{n}$; and per 47 Euc. I.

$\text{CG} = \dfrac{q\sqrt{(nn + dd)}}{n}$; again, GB $:$ GP $::$ AB $:$ AN $= \dfrac{2da}{\sqrt{(dd + nn)}}$,

and AG $:$ AN $::$ CG $:$ $cc = \dfrac{2daq}{n\sqrt{(d^2 + m^2)}}$; hence the solidities,

of the cone $\text{ACB} = \dfrac{4a^3sd}{3n}$, of $\text{c}g\text{G} = \dfrac{4q^3sd}{3n}$, and the scalenous $\text{CAG} = $

$\dfrac{4sdaq\sqrt{aq}}{3n}$; whence the solidity of the greater hoof $\text{AGB} = \dfrac{4asd}{3n}$

$(aa - q\sqrt{aq})$, and lesser $\dfrac{4sdq}{3n}(a\sqrt{aq} - qq)$. Or putting the frustum's height $=$ A, greater diameter $=$ D, lesser $= d$, diff. $= x$, mean proportional between them $=$ B and $\cdot78539 = s$; then

Greater hoof $= \dfrac{\text{D}As}{3x}(\text{DD} - \text{B}d) = 48741\cdot05$.

Lesser hoof $= \dfrac{d\text{A}s}{3x}(\text{DB} - dd) = 17232\cdot55$, &c.

VII.   QUESTION 241, *by Mr.* Will. Daniel.

It is universally agreed, that the heat at any moment of time, on any day, is proportionable to the rectangle made of the sine of the sun's altitude, and the arc of time expressing his continuance above the horizon: which being allowed, it is required to find what time of the day will be the hottest at Coventry, (lat. 52° 30′) on August 14, 1743.

N. B. This is one of the two problems which Mr. Stone (in his translation of L'Hospital's Fluxions) challenges the mathematicians of Europe to answer.

Answered by Mr. J. Turner, of York.

Let $b =$ length of the semi-diurnal arch $= 1\cdot8233$; $x =$ arch of the hour from noon, its cosine $= 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720}$ &c. $d =$ rectangle of the sines of the lat. and declination $= \cdot1493$; $c =$ $\cdot5979$ the rectangle of their cosines; then $c \times : 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} -$ $\dfrac{x^6}{720}$ &c. $+ d =$ sine sun's altitude; and $(cb + cx) \times : 1 - \dfrac{x^2}{2} +$ $\dfrac{x^4}{24} - \dfrac{x^6}{720}$ &c. $+ db + dx$ by the question must be a maximum; which put into fluxions and reduced, gives $cbx + \dfrac{3cx^2}{2} - \dfrac{cbx^3}{6} -$ $\dfrac{5cx^4}{24} + \dfrac{cbx^5}{120}$ &c. $= c + d$; or $1\cdot09015x + \cdot8968x^2 - \cdot1817x^3 -$ $\cdot1246x^4 + \cdot0991x^5$ &c. $= \cdot7472$. Here $x = \cdot5049 =$ an arch of $28° 56'$, so the time is $1^h 55^m 44^s$ post meridian.

An answer by Mr. Thacker.

Put radius $= 1$; d and c for the sine and cosine of the sun's declination; b and $a =$ sine and cosine of the latitude given; x and y the sine and cosine of the hour from noon; z the arc itself: then by spherics $db + cay =$ sine of BO the sine of the sun's altitude; which being multiplied by $n + z$ ($n =$ semi-d. arc) the arc of time passed over from sun rising, gives $(db + cay)(n + z)$; which by the question is proportionable to the

heat, and must therefore be a maximum; put into fluxions, is $ca\dot{y} \times$ $(n + z) + \dot{z}(db + cay) = 0$; but by the nature of the circle $x : 1$ $:: -\dot{y} : \dot{z}, \therefore - \dfrac{\dot{y}}{x} = \dot{z}$; which put for $\dot{z}$ is $ca\dot{y}(n + z) - \dfrac{\dot{y}}{x} \times$ $(ab + cay) = 0$; and therefore $ca\dot{y}(n + z) = \dfrac{\dot{y}}{x}(db + cay)$, which divided by $\dot{y}$ gives $ca(n + z) = \dfrac{1}{x}(db + cay)$; this divided by ca will make $n + z = \dfrac{1}{x}\left(\dfrac{db}{ca} + y\right)$.
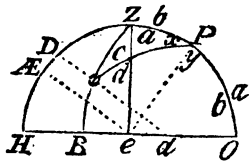
From which equation it appears that the hottest time of the day is, when the arc of time passed over from sun-rising is equal to the

rectangle of the tangents of the sun's declination and latitude, plus the cosine of the hour or arc from noon, drawn into the co-secant of the arc passed over since noon. In numbers, the tangent of sun's declin. $10° 53' = 9.2839070$ multiplied by tangent lat. $52° 30' =$ log. 10.1150195 gives $9.3989265 =$ cosine $75° 29' 40''$ the arc of time from midnight, which subtracted from 180 leaves $104° 30' 20''$ the semi-diurnal arc $= 12$; to which add $28° 56'$ time after noon makes $133° 26' 20''$. This in decimals of degrees, multiplied by 0.174532, &c. (i. e. 360) 6.28318 $(.01745)$ gives the arc of the circle 2.3263535. And tang. decl. $\times$ tang. lat. $= .2505642$, plus cos. arc $28° 56' = .8751832$ from noon ($= n + z$) is 1.1257474 multiplied by the co-secant $28° 56' = 2.0670056$ gives 2.326 &c. equal to the other side of the equation, which is proved right. But the arc from noon is found thus: If v be put for the number of degrees passed over since noon; then will $.017453 \times (104° 30' + v)$ $=$ co-secant $v \times (.2505642 + $ cos. $v)$; whence by a few trials, v may be found $= 28° 57'$, which in time is $1^h 55^m 48^s$ afternoon.

If any one should dislike this guessing method, the following may be thought more valuable, $viz.$

Put $t =$ tangent of half the time from noon; then will $\dfrac{2t}{1 + tt}$

$= x$; and $\dfrac{1 - tt}{1 + tt} = y$; also, by the nature of the circle, $z = 2t -$

$\dfrac{2t^3}{3} + \dfrac{2t^5}{5} - \dfrac{2t^7}{7}$, &c. whence $n + z = \dfrac{1}{x}\left(\dfrac{db}{ca} + y\right)$ is $n + 2t -$

$\dfrac{2t^3}{3} + \dfrac{2t^5}{5}$ &c. equal to $\dfrac{1 + tt}{2t} \times \dfrac{db}{ca} \times \dfrac{1 - tt}{1 + tt} + 1 - tt$; and

the equation ordered

$$\left.\begin{array}{c} + 2n \\ - 2 \end{array}\right\} t \left.\begin{array}{c} + 4 \\ - \dfrac{bd}{ca} \end{array}\right\} t^2 + 2t^3 - \dfrac{4t^4}{3} = \dfrac{4t^6}{5} + \dfrac{4t^8}{7} + \dfrac{4t^{10}}{9}, \quad \&c. =$$

$\dfrac{db}{ca}$; which series reverted will give the value of $t =$ tangent $14° 28'.$

VIII. QUESTION 242, by Mr. J. May, $jun.$

Last spring, 1742, being at sea in north latitude, we had great storms for several days together, succeeded with cloudy weather, which hindered our making observations; at last when it cleared up, with Mr. Hadley's octant, upon deck, I endeavoured to take the sun's meridian altitude, but unfortunately thick clouds prevented it. Now being at a great loss to know where we were, we endeavoured to contrive some other way; and accordingly waiting a few minutes, it cleared up again, and we took the sun's altitude (after allowance for refraction and dip of horizon, &c.) $57° 24' 52''$; tarrying 26

minutes by a good watch, we observed the sun's height was 55° 35′ 19″; and 25 minutes after this, the sun's altitude was 53° 16′ 15″. From whence is required the latitude we were in, the time of each observation, and the sun's declination, by a general theorem for all problems of this nature?

<p align="center">*Answered by Mr.* T. Cowper.</p>

Let c = cosine of z1 = 57° 24′ 52″; t = cosine of z2 = 55° 35′ 19″; d = cosine of z3 = 53° 16′ 15″; h = sine of 1P2 = 6° 3′, its cosine = p; m = sine of 1P3 = 12° 45′, n = its cosine; x = sine DP1, y = its cosine; then $py - hx$ = cosine ZP2; $ny - mx$ = cosine ZP3. Then by theorem 1, in the solution to quest. 222, we have
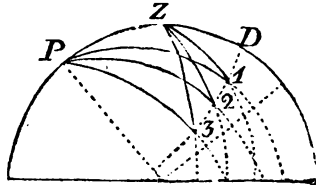
$$\frac{c + ty - t - cpy + chx}{y - py + hx} = \text{cosine ZD, and}$$

$$\frac{c + dy - d - cny + cmx}{y - ny + mx} = \text{cosine ZD; which transposed and reduced}$$

is $cpyy + tnyy + dyy - dpyy - cnyy - tyy + chx + tmx - dhx - cmx = cpy + tny + dy - dpy - cny - ty + chxy + tmxy - dhxy - cmxy.$ Put $r = cp + tn + d - dp - cn - t$, or $= \cdot 0001692$; and $s = ch + tm - dh - cm$, or $= \cdot 00077322$; then the equation above, after substitution, will stand, $ryy + sx = ry + syx$, i. e. $sx - sxy = ry - ryy$, and dividing by $1 - y$, we have $sx = ry$;

$$\therefore \frac{x}{y} = \frac{r}{s} \text{ the tangent of } 12° 20′ 36″:$$ consequently the 1st observation was 12ʰ 49ᵐ 22ˢ, the 2d at 1ʰ 15ᵐ 22ˢ, the 3d at 1ʰ 40ᵐ 20ˢ P.M. And by the 1st and 2d theorem in the same 222d quest. is found the sun's meridian altitude = 58° 15′ 57″, his septentrional depression = 17° 20′ 35″; hence the latitude is 51° 58′ 49″, and ☉'s declination = 20° 45′ 36″ north.

<p align="center">*The same answered by* F. R. S.</p>

Let the quantities be represented as above; also let r and s = sine and cosine of the latitude; and a and c the sine and cosine of the sun's distance from the pole. Then in the three triangles by the spheric theorem, we have

<p align="center">
(1) $er + asy = c$,
(2) $er + aspy - ashx = t$,
(3) $er + asny - asmx = d$.
</p>

In the (1) $er = c - asy$, which substituted in the other two, make
(4) $c - asy + aspy - ashx = t$, (5) $c - asy + asny - asmx = d$;

per (4) $- asy + aspy - ashx = t - c = g$, $\therefore as = \dfrac{g}{-y + py - hx}$

$= \dfrac{g}{-y(1-p) - hx}$: call $1 - p = v$ = versed sine 6° 30′, then

will $as = \dfrac{g}{-yv - hx}$; by the (5) $- asy + asny - asmx = d - c$

$= k$, $as = \dfrac{\cdot k}{-y + ny - mx} = \dfrac{k}{-y(1-n) - mx}$; and if $1 - $

$n = w$ = vers. sine 12° 45′, then will $as = \dfrac{k}{-yw - mx}$; conse-

quently $\dfrac{g}{-yv - hx} = \dfrac{k}{-yw - mx}$, or $\dfrac{g}{yv + hx} = \dfrac{k}{yw + mx}$, which

out of fractions $gyw + gmx = kyv + kpx$, $\therefore \dfrac{x}{y} = \dfrac{kv - gw}{gm - kh} = $

tangent of the $\angle$ DP1 $= 12°\ 20'\ 36'' = $ time $49^m\ 22^s$. Whence the
answer will come out as above.

I cannot but be persuaded that this curious problem will point
out a way to be very useful in navigation for determining the longi-
tudes as well as latitudes : for, supposing at sea we know neither,
or had not the time of the day, but were furnished with a quadrant
to take altitudes of the sun, and could find the difference in time
between each observation, which a common pocket watch with a mi-
nute hand would give us very well, or with a second hand better.
For though that watch or a clock was incapable of keeping true time
at sea, yet it might very well measure a few minutes between one ob-
servation and another, in which space the error must be very incon-
siderable. Now when this was done, by solving this problem, we get
the latitude and the true times of the day, and then it would be no
very difficult task to rectify the longitude pretty near. In order to
this we shall deduce a theorem in words, by which any one that is but
skilled in the common cases in trigonometry may put it in practice.

A Theorem for the Hour of the Day.

1. The difference between the first and second altitude, drawn
into (i. e. multiplied by) the versed sine of the arch of time between
the first and second observation ; made less by the difference between
the first and second altitude, multiplied into the versed sine of the
arch of time between the first and third observations.

2. The difference between the first and second altitude, drawn into
the right sine of the arch of time, between the first and third observa-
tions ; minus the difference between the first and third altitude,

drawn into the right sine of the arch of time between the first and second observations.

Lastly, Divide the former difference by the latter, and the quotient will be the tangent of the arch of the time from noon.

The same answered by Mr. J. May, the Proposer.

Put $a =$ sine $57° 24' 52''$, $b =$ that of $55° 35' 19''$, $c =$ that of $53° 15' 16''$, the sine of 26^m or $6° 30' = m$, cos. $= n$, $f =$ sine $12° 42' (= 51^m)$ the time between the first and last observ. its cos. $= g$, radius $= 1$. Then put $a - b = h$, $a - c = i$, $r - n = s$, $r - g = t$; then the tangent of the hour angle from noon when the greatest altitude was taken will be $= \dfrac{ht - is}{mi - hf}$, $r = ·2190328 = 12°$ $21' 16''$, which in time is $49^m 25^s$, or after 12 o'clock; the second $1^h 15^m 25^s$; and the last at $1^h 40^m 25^s$, according to the altitude's decrease.

Now put the cosine of $12° 21' 16'' = d$; of $25° 6' 16''$ (the arc of $1^h 40^m 25^s$ time of the last observation) $= e$; put likewise $d - e = q$, $r - e = l$, and $r + d = p$.

Then the sine of the sun's southern altitude will be $= b + h ÷ q$ (see his answer to quest. 222, Diary 1742) the degrees of which put $= w$. Likewise the sine of the sun's depression in the north will be $pi ÷ q - a$; the degrees of it put $= u$; then the sun's declination will be $\frac{1}{2}(w + u) = 20° 47'$ nearly; and the cosine of the latitude $\frac{1}{2}(w - u) = 38° 4' 45''$, and $51° 55' 15''$ the latitude required.

IX.   QUESTION 243, by Mr. John Powle.

Let there be three spherical and perfectly elastic bodies, A, B, C: the weight of A $= 3$ pounds, and of C $= 27$. Now it is required to find the weight of the intermediate body B, so that A striking B at rest, and B, with the motion acquired by the stroke, striking C at rest, the motion produced in C, shall be a maximum?

Answered by Mr. J. Landen.

The bodies are A, B, C. $\left.\right\}$ Then according to Mr. Keil's Introduct.
Their weights 3, x, 27.

Phys. we have $\dfrac{2a}{x + a} =$ the celerity wherewith the body B will approach C, and $\dfrac{4xa}{xx + xa + xc + ac} =$ the velocity of C after the impulse; the fluxion of which being made $= 0$, and reduced, we have $x = 9$, a mean proportional between A and C.

Mr. Tho. Cowper's *answer.*

The bodies and weights denoted as above, and putting 1 to express the velocity of A: from Dr. Keil's demonstration about the motions of elastic bodies is deduced this analogy: As the sum of the bodies : twice the weight of the moving or striking body :: the velocity of the striking body before percussion: the velocity of the quiescent body after it. That is, $x + a : 2a :: 1 : \dfrac{2a}{x + a}$ = velocity of B after the stroke. Again, $x + c : 2x ::$

$\dfrac{2a}{x + a} : \dfrac{4ax}{xx + cx + ax + ca}$ = the celerity of c after the stroke; which, per question, is a maximum, and the fluxion thereof $4ax^2\dot{x} + 4acx\dot{x} + 4aax\dot{x} - 8ax^2\dot{x} - 4acx\dot{x} - 4aax\dot{x} = 0$. Reduced, gives $x = \sqrt{ac} = 9$ pounds.

THE PRIZE QUESTION, *by the late illustrious Sir* I. Newton.

Three staves being erected, or set up on end, in some certain place of the earth, perpendicular to the plane of the horizon, in the points A, B, and c; whereof that which is at A, is 6 feet long; that in B, 18; that in c, 8; the line AB being 33 feet long: It happens on a certain day in the year, that the end of the shadow of the staff A passes through the points B and c: and of the staff B, through A and c; and of the staff c, through the point A.

To find the sun's declination, and the elevation of the pole or day, and the place where this shall happen.

Note, this is the 42d problem in Sir Isaac Newton's Universal Arithmetic; and it may seem a piece of vanity in attempting to give a solution after the greatest of men; but having in the winter 1740, taken a great deal of pains to bring out a solution, and never being able to get his numbers for the declination and latitude precisely the same, I was fond to think his were exact, and wrought it over and over again; at first it came out an adfected equation, then a quadratic, and at last happily by a simple equation; and having taking the pains to prove all the numbers (not depending on the logarithms) found them agree in every particular, and by construction to form a true conic section. We therefore humbly presume, that in a calculus so prolix and difficult (in Sir Isaac's method) there might happen a small error, or at least some press fault of the editions, or in the translation; which we hope to make more fully appear in the next year's Diary.

Y 4

Answered by Mr. Ant. Thacker.

Put $a = 6$, $b = 18$, $c = 8$ feet, the height of the three staves at A, B, C; the line AB $= 33$ feet $= d$; the distance between the top of the staff A and bottom of the staff B ($= a$B) call e; the distance between the top of the staff B and bottom of staff A ($= b$A) call f; then

will $\dfrac{(c + a)\,(e + f)}{2b + 2a} + \dfrac{(b + a)\,(c - a)}{2e + 2f} = m = 21{\cdot}0789$, the dis-

tance from the top of C to the bottom of A. Then (by 47 Euc. I.)

AC is found $= 19{\cdot}501$, which call x; also $\dfrac{(c + b)\,(e + f)}{2b + 2a} +$

$\dfrac{(b - c)\,(b + a)}{2e + 2f} = n = 40{\cdot}216$, the length between the top of B and

bottom of C; and hence CB is found $= 35{\cdot}962 = z$. Now putting

$\dfrac{be - fa}{df + de} = h = 16111$; $\sqrt{(xx + aa)} = s = 20{\cdot}404$; $\dfrac{cs - am}{xm + xs} =$

$k = {\cdot}0454$; and $\dfrac{dd + xv - zz}{2dx} = v = {\cdot}13501$; then will

$\dfrac{2 - 2vv}{1 + hh + kk - 2hkv - vv} = 1{\cdot}948315$, the versed sine of an arc

which is double the latitude $161°\,30'$; whose half is $= 80°\,45'$, the true latitude required.

Now calling the sine of the latitude p, then will $\dfrac{pc + pa}{s + m} =$

${\cdot}33309 =$ the sine of $19°\,27'$, the sun's declination north.

Sir I. Newton, the author of this problem finds the lat. $80°\,45'$ $20''$, and declination $19°\,27'\,20''$, as may be seen in the 42d prob. of his Universal Arithmetic, in the English edit. 1720, p. 151, where the translator Mr. Raphson, Mr. Cunn, or the printer, committed a blunder, making the line AB $= 30$, instead of 33. By reason of which disappointment, the solution is here shorter than was designed; but the investigation of the theorem above we have printed in the first volume of Diary Questions [Thacker's Miscel.] If the line AB was $= 30$, then the latitude is $80°\,4'\,7''$, and declination $21°\,7'\,4''$
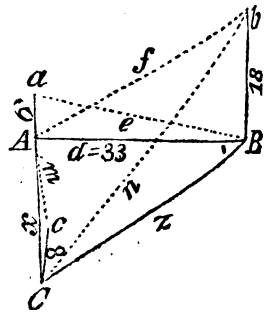
Here follows the Solution from Thacker's *Miscellany alluded to above.*

Call height of the staff in the point A, (a); that in B, (b); that in c, (c); the line AB (d); the distance betwixt the top of the staff in A, and the bottom of the staff in B, (e); the distance betwixt the top of the staff in B, and the bottom of the staff in A, (f); and let AC $= x$, CB $= z$; then $\sqrt{(xx + aa)} = s$, and $\sqrt{(xx + cc)} = m$, the distances betwixt the top of the staff in A and bottom of the staff in c, and top of the staff in c and bottom of the staff in A, respectively. And suppose NS the meridian line, radius $= 1$; then will $a \div e$ and $d \div e$ be the sine and cosine of the sun's altitude when he makes the shade AB by the staff A ; for the cosine of the angle ABN (or BAN) $=$ cosine of the sun's azimuth from the north at that time, put $\mp v$. Again, $b \div f$ and $d \div f$ express the sine and cosine of the sun's altitude when he makes the shade BA by the staff B; the cosine of BAN (or ABN) $=$ cosine of the sun's azimuth from the north at that time $\mp v$. Also $a \div s$ and $x \div s$ will be the sine and cosine of the sun's altitude when he makes the shade AC by the staff A; and let $\mp y$ be put to denote the cosine of the angle ACN (or CAN $=$) the sun's azimuth from the north at that time; again, $c \div m$ and $x \div m$ will express the sine and cosine of the sun's altitude when he makes the shade CA by the staff c ; and $\mp y$ the cosine of the angle CAN (or ACN) $=$ the cosine of the sun's azimuth at that time from the north. Lastly, let p and q be put to express the sine and cosine of the required latitude; and g, the sine of the sun's declination. Then (1) $\dfrac{pa}{e} \mp \dfrac{qdv}{e} = g$; (2) $\dfrac{pb}{f} \mp \dfrac{qdv}{f} = g$; (3) $\dfrac{pa}{s} \mp \dfrac{qxy}{s} = g$; (4) $\dfrac{pc}{m} - \dfrac{qxy}{m} = g$. Whence, by equation (1), we have $\pm qdv = ge - pa$; and by equation (2) $pb - gf = + qdv$; consequently $ge - pa = pb -gf$; and $g = \dfrac{pb + pa}{e + f}$; again, by equation (3), we get $\mp qxy = gs - pa$, and by equation (4), $pc - gm = + qxy$: consequently $gs - pa = pc - gm$, $\therefore g = \dfrac{pc + pa}{s + m}$; therefore
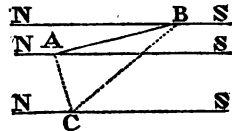
$$\frac{pc + pa}{s + m} = \frac{pb + pa}{e + f}, \text{ and } \frac{c + a}{s + m} = \frac{b + a}{e + f}; \text{ or } \frac{(c + a)(e + f)}{b + a}$$

$= s + m$; $\therefore \dfrac{(c + a)(e + f)}{b + a} - m = s$, which squared is

$$\frac{(c+a)^2 (e+f)^2}{(b+a)^2} - \frac{2m(c+a) (e+f)}{b+a} + m^2 = s^2:$$ but as

$\sqrt{(x^2 + a^2)} = s$, and $\sqrt{(x^2 + c^2)} = m$, $\therefore (x^2 =) s^2 - a^2 = m^2 - c^2$, $\therefore s^2 = m^2 + a^2 - c^2$, which put for s^2 in the above equation,

makes $\dfrac{(c+a)^2 (e+f)^2}{(b+a)^2} - \dfrac{2m(c+a) (e+f)}{b+a} + m^2 = m^2 + a^2$

$- c^2$, $\therefore$ by transposition, &c. we have $\dfrac{(c+a) (e+f)}{2b+2a} +$

$\dfrac{(b+a) (c-a)}{2e+2f} = m$ the distance betwixt the top of the staff in c

and bottom of the staff in A, and so AC is found.

And by the very same way that m was found, the distance betwixt the top of the staff in B and bottom of the staff in c may be found, and if n be put for it, the equation will stand $\dfrac{(c+b) (e+f)}{2b+2a} +$

$\dfrac{(b-c) (b+a)}{2c+2f} = n$, and therefore cB is known. For the other part of the analysis, we have, by the help of equations (1)

and (2) $\dfrac{pa}{e} \pm \dfrac{qdv}{e} = \dfrac{pb}{f} \mp \dfrac{qdv}{f}$ $\therefore \pm \dfrac{qdv}{e} \pm \dfrac{qdv}{f} = \dfrac{pb}{f} - \dfrac{pa}{e}$ $\therefore$

$\pm v = \dfrac{(be - fa)p}{(df + de)q}$; put $\dfrac{be - fa}{df + de} = h$, then will $\pm v = \dfrac{hp}{q}$, whose

square is $v^2 = \dfrac{h^2 p^2}{q^2}$ $\therefore \sqrt{(1 - v^2)} = \sqrt{\left(1 - \dfrac{h^2 p^2}{q^2}\right)} =$ the sine of

the angle BAS. And by the same way of reasoning, from equations

(3) and (4), is found $\pm y = \dfrac{(cs - am)p}{(xm + xs)q}$; put $\dfrac{cs - am}{xm + xs} = k$, then

will $\pm y = \dfrac{kp}{q}$, which squared makes $y^2 = \dfrac{k^2 p^2}{q^2}$ $\therefore \sqrt{(1 - y^2)} =$

$\sqrt{\left(1 - \dfrac{k^2 p^2}{q^2}\right)}$, the sine of the angle CAS.

Now having all the three sides of the triangle ABC given, we find the angle CAB to be less than a right angle; therefore the cosine belonging to the sum of the angles BAS and CAS is $\dfrac{hkp^2}{q^2} - \sqrt{(1 - }$

$\dfrac{h^2 p^2}{q^2})\sqrt{\left(1 - \dfrac{k^2 p^2}{q^2}\right)}$. But from the triangle we have $\dfrac{x^2 + d^2 - z^2}{2dx} =$

v the cosine of the angle CAB, consequently $\dfrac{hkp^2}{q^2} - \sqrt{\left(1 - \dfrac{k^2 p^2}{q^2}\right)}$

$\sqrt{\left(1 - \dfrac{k^2 p^2}{q^2}\right)} = v$ $\therefore \dfrac{hkp^2}{q^2} - v = \sqrt{\left(1 - \dfrac{h^2 p^2}{q^2}\right)}\sqrt{\left(1 - \dfrac{k^2 p^2}{q^2}\right)}$ which

squared, &c. makes $- \dfrac{2khvp^2}{q^2} + v^2 = 1 - \dfrac{h^2p^2}{q^2} - \dfrac{k^2p^2}{q^2}$, or, by

writing $1 - p^2$ for q^2, $- \dfrac{2hkvp^2}{1-p^2} + v^2 = 1 - \dfrac{h^2p^2}{1-p^2} - \dfrac{k^2p^2}{1-p^2}$.

And by reduction $2p^2 = \dfrac{2 - v^2}{1 + h^2 + k^2 - 2hkv - v^2}$ the versed sine

of an arc, which is double the required latitude. And the sine of the

sun's declination is known to be $g = \dfrac{nc + pa}{s + m}$.

Another Solution, by Mr. Skene, taken from Davis's Mathematical Companion for 1805.

Among a variety of problems, which, were proposed, about the time of Des Cartes, as instances of the excellence of the modern analysis, and its superiority to the ancient, the following appears to be the most abstruse, as well as the most curious and remarkable. Des Cartes * speaks of it in his Epistles as a problem of the greatest difficulty, and eminently fitted ad notandam industriam bene disce-rendi equationes. From Francis à Schooten we learn that it was first published in the year 1640, in a very ingenious book, entitled Den Onwissen Wis-konstenaér I. I. Stampioënius.† The solution there exhibited, is revised, corrected, and improved in Schooten's Addi-tamentum to his Commentary on Des Cartes's Geometry.‡ The Ad-ditamentum, indeed, is principally occupied with this problem, which is placed at the end of the Commentary as a proof, as he himself expresses it, non facile problema aliquod datum iri, quod hanc geometriam affugiat, aut ejusdem methodo solvi non possit. Sir Isaac Newton has likewise given a solution to this problem in his Arithmetica Universalis, published by Whiston 1707. It may be justly concluded, that it would be in vain to seek for a more ele-gant solution of a problem which has passed through the hands of such eminent men. Newton's, in particular, is one of the finest specimens of algebraic analysis in the whole circle of the mathematical sciences. But as the question has been supposed to be beyond the reach of the ancient geometry, I shall make no apology for giving the following solution, which is effected without the assistance of algebra.——The problem is thus enunciated by Des Cartes.——
" Tres baculi erecti sunt ad perpendiculum in horizontali plano, ex punctis A, B, c ; et baculus A est 6 pedum, B 18 pedum, c 8 pe-dum, et linea AB est 33 pedum ; et una atque eadem die extremitas umbræ solaris, quam facit baculus A, transit per puncta B & c, ex-tremitas umbræ baculi B per A & c, et ex consequenti etiam baculi c per A & B. Quæritur in quænam poli altitudine, & qua die anni it

* Ren Des Cartes, Epist. Part ii. Epist. lxxi. page 257, Francf. Ed.
† Edited by Jacob à Waessenaer. l.
‡ Schooten merely premised a lemma and added an explanation by Erasmius Bartholinus. l.

contingat." This enunciation is in part erroneous, for it does not
follow, as is here affirmed, that the end of the staff at c must pass
through the points A and B, because the end of the staff at A passes
through B and C, and that of the staff at B through A and C: ano-
ther circumstance must be given, namely, that the end of the staff
at C passes through A, so that it is necessary to change " et ex con-
sequenti, &c. into baculi c per punctum A, et ex consequenti etiam
baculi c per punctum B." In this manner it is enunciated by Schoo-
ten, with this difference only, that the consequence from the other
part of the data the end of the staff at c's passing through B is de-
monstrated by him algebraically; and the same is done by Newton.

The following is a geometrical demonstration.

The end of the shadow of the staff at A describes a conic section,
which let be CEDGB (*fig.* 1.). Let A, B, C, be the
places of the three staves, and let BA, CA be produced
to meet the curve in D, G. Draw also EF through
A parallel to BC, to meet the curve in E, F, and
join DE, GF. Then when A casts the shadow AD,
B casts the shadow BA; and when A casts AE, B
casts BC. Therefore AD is to BA as AE to BC;
but AE is parallel to BC by the construction, con-
sequently DE is parallel to AC. Now BC being pa-
rallel to EF, CG parallel to ED, and c, E two points
in the conic section, GF is parallel to BD, by Simson's Con. Sect.
lib. iv. prop. 29. Hence BC is to AF as CA to AG, and the end of the
shadow of c therefore passes through the point B, whatever be the
conic section described by the extremities of the shadows. *Q. E. D.*
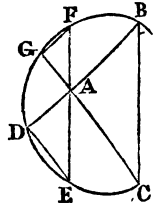
This property in the ellipse may be still more easily demonstrated
by first shewing its truth in the circle, and thence transferring it to
the ellipse by cutting a cylinder obliquely.

We shall now proceed to point out our method of resolving the
problem, for which purpose the following lemma is required.

Lemma. If the altitudes of the sun be taken at the same place on
the same day, when he is in two opposite directions, the sum of
their tangents will be to the sum of their secants as the sine of the
sun's declination to the sine of the latitude of the place.

Let z (*fig.* 2.) be the zenith, P the pole, ZP the co-latitude, ZQ,
ZO the co-altitudes, and PQ or PO the comple-
ment of the sun's declination. Draw PM per-
pendicular to ZO, and QM, OM will be equal
to each other; therefore OM is equal $\frac{1}{2}$(ZO
+ ZQ) and ZM equal to $\frac{1}{2}$(ZO — ZQ). Then,
by an elementary property of spherical tri-
angles, cos PQ : cos ZP :: cos OM : cos ZM
:: cos $\frac{1}{2}$(ZO + ZQ) : cos $\frac{1}{2}$(ZO — ZQ). But
cos $\frac{1}{2}$(ZO + ZQ) is to the cos $\frac{1}{2}$(ZO — ZQ), as
the cot ZO + cot ZQ to the cosec ZO + cosec
ZQ; therefore cos PQ : cos ZP :: cot ZO +
cot ZQ : cosec ZO + cosec ZQ, that is, the sine of the sun's decli-

nation is to the sine of the latitude of the place as the sum, &c.
Q. E. D.
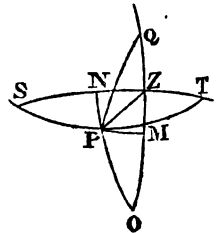
Cor. If two altitudes of the sun be taken on the same day, when
he is in opposite directions, and other two altitudes when he is like-
wise in opposite directions, the sum of the tangents of the first two
will be to the sum of the tangents of the second two, as the sum of
the secants of the first two to the sum of the secants of the second
two. This corollary follows directly from the lemma, as the latitude
and declination remain the same.

Now as the heights of the two staves at A and B (*fig* 3.) are given, and
the lengths of their respective shadows, when the
sun is in opposite directions, the two altitudes
are likewise given, and consequently the ratio
of the sum of their tangents to the sum of their
secants. Let AD represent the staff at A, and
CE the staff at C. Take CF on EC produced
equal to AD, and join AE, AF. Then will CE
and CF represent the tangents, and AE, AF the
secants of the sun's altitudes, when in the op-
posite directions AC, CA; and therefore by the
lemma the ratio EF to AE + AF is given; but
EF is given, consequently AE + AF is given.
Now, by an elementary proposition in plain trigonometry, AE + AF is
to EF as the difference between CE, CF to the difference between AE, AF;
which difference is therefore given, and consequently AE, AF. Hence
AC, the distance between the two staves A and C, becomes known.
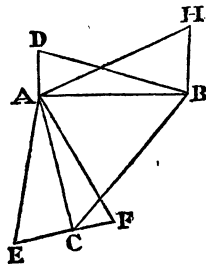
In a manner exactly similar may the distance BC, between the two
staves B and C, be found, for the data are the same in both.

Then having given the three sides of the triangle ABC, the angle
BAC may be determined ; and as the heights of the staves at A and C
are given, and their distance AC, the sun's altitudes, when in the op-
posite directions CA, AC, are likewise given.

Now let z on the globe represent the zenith of the place, (*fig*. 2.)
at which the observation is made, P the pole, ZP the co-latitude, ZQ,
ZO the zenith distances of the sun, when in the opposite directions
AB, BA, and ZS, ZT the zenith distances of the sun, when in the op-
posite directions CA, AC. Draw the great circles PQ, PO, PS, PT,
which will be all equal, being the co-declination of the sun : also PM
perpendicular to OQ, and PN to ST.

It is evident that ZM, which is equal to $\frac{1}{2}$(ZO — ZQ), and ZN,
which is equal to $\frac{1}{2}$(ZS — ZT), are both given, but NZM is equal to
the angle BAC (*fig*. 3.), being the difference of the sun's azimuths
when in the directions AB, AC. Hence this construction.

Take ZM in ZO equal to $\frac{1}{2}$(ZO — ZQ), and ZN in ZS, whose position
is given, equal to $\frac{1}{2}$(ZS — ZT); draw the great circle MP perpendi-
cular to OQ, and NP perpendicular to ST; the point P where they inter-
sect each other is the pole, which being determined the latitude and
declination are given.

We shall now give the numerical calculation.

Let AD *(fig. 3.)* perpendicular to AB be equal to the staff at A, BH perpendicular to AB equal to the staff at B, and let BD, AH be drawn. Then will AD $= 6$, BH $= 18$, CE $= 8$, and AB $= 33$. Hence the tan ABD $= \dfrac{2}{11}$, and sec ABD $= \dfrac{5\sqrt{5}}{11}$, also tan BAH $= \dfrac{6}{11}$ and

sec DAH $= \dfrac{\sqrt{157}}{11}$. Therefore $\dfrac{2}{11} + \dfrac{6}{11} : \dfrac{\sqrt{125}}{11} + \dfrac{\sqrt{157}}{11} :: 8 :$

$\sqrt{125} + \sqrt{157} :: 1 : 2\cdot 963788 ::$ sum of the tangents of the sun's altitudes : sum of their secants, when in any two opposite directions. Hence, EF being $= 14$, we get AE $= 21\cdot 0839221$, AF $= 20\cdot 4091099$, and AC $= 19\cdot 5072236$. In like manner, we find BC $= 35\cdot 963158$. Now in the triangle ABC the three sides are given, from which the angle BAC is easily computed $= 82° 8' 5''$. We have also given *(fig. 2.)*, ZO $= 79° 41' 43''$, ZQ $= 61° 23' 22''$, ZS $= 72° 54' 11''$, ZT $= 67° 42' 1''$, and the angle SZO $=$ MZN $= 82° 8' 5''$; hence ZM $= 9° 9' 10\frac{1}{2}''$, ZN $= 2° 36' 5''$, and by spherics cos PZM : cos PZN :: tan ZM : tan ZN : but the sum of the angles PZM, PZN is given $= 82° 8' 5''$, and the ratio of their cosines, from which we easily find PZM $= 8° 20' 8''$, and PZN $= 73° 47' 57''$. Whence the latitude and declination may be computed by the following analogies, tan ZM : rad :: cos PZM : cotan ZP $=$ tan lat. and by the lemma $1 : 0\cdot 3374061 ::$ sin lat. : sine declination. In this manner, we find that the latitude of the place was $80° 45' 4''$, and the sun's declination $19° 27' 9''$, which numbers agree exactly with those obtained in so different a manner by Francis Schooten and Sir Isaac Newton.

Newton's *Solution being so very elegant and complete, the Editor has thought fit to insert it here for the sake of those who are not in possession of the Arithmetica Universalis.*

    Solution by Sir Isaac Newton.

Because the shadow of each staff describes a conic section, or the section of a luminous cone, whose vertex is the top of the staff; I will feign BCDEF to be such a curve, (whether it be an hyperbola, parabola, or ellipse) as the shadow of the staff A describes that day, by putting AD, AE, AF, to have been its shadows, when BC, BA, CA, were respectively the shadows of the staves B and C. And besides I will suppose PAQ to be the meridional line, or the axis of this curve, to which the perpendiculars BM, CH, DK, EN, and FL, being let fall, are ordinates. And I will denote these ordinates indefinitely by the
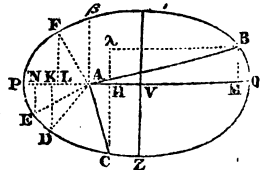
letter y, and the intercepted parts of the axis AM, AM, AK, AN, and AL by the letter x. I will suppose, lastly, the equation $aa \perp bx \perp cxx = yy$, to express the relation of x and y, (*i. e.* the nature of the curve) assuming aa, b and c, as known quantities, as they will be found to be from the analysis. Where I made the unknown quantities of two dimensions only because the equation is to express a conic section: and I omitted the odd dimensions of y, because it is an ordinate to the axis. And I denoted the signs of b and c, as being indeterminate by the note $\perp$, which I use indifferently for $+$ or $-$, and its opposite $\top$ for the contrary. But I made the sign of the square aa affirmative, because the concave part of the curve necessarily contains the staff A, projecting its shadows to the opposite parts (C and F, D and E); and therefore if at the point A you erect the perpendicular Aβ, this will somewhere meet the curve, suppose in β, that is, the ordinate y, where x is nothing, will still be real. From thence it follows that its square, which in that case is aa, will be affirmative.*

It is manifest therefore, that this fictitious equation $aa \perp bx \perp cxx = yy$, as it is not filled with superfluous terms, so neither is it more restrained than what is capable of satisfying all the conditions of this problem, and will denote the hyperbola, ellipse, or parabola, according as the values of aa, b, c, shall be determined, or perhaps found to be nothing. But what may be their values, and with what signs b and c are to be affected, and thence what sort of a curve this may be, will be manifest from the following analysis.

The former Part of the Analysis.

Since the shadows are as the altitudes of the Staves, you will have BC : AD :: AB : AE (:: 18 : 6) :: 3 : 1. Also CA : AF (:: 8 : 6) :: 4 : 3. Wherefore naming AM $= r$, MB $= s$, AH $= t$, and HC $= \perp v$. From the similitude of the triangles AMB, ANE, and AHC, ALF, AN will be $= -\frac{1}{3}r$. NE $= -\frac{1}{3}s$. AL $= -\frac{3}{4}t$, and LF $= \top \frac{3}{4}v$; whose signs I put contrary to the signs of AM, MB, AH, HC, because they tend contrary ways with respect to the point A from which they are drawn, or to the axis PQ on which they stand. Now these being respectively written for x and y in the fictitious equation $aa \perp bx \perp cxx = yy$.

r and s will give $aa \perp br \perp crr = ss$.

$-\frac{1}{3}r$ and $-\frac{1}{3}s$ will give $aa \top \frac{1}{3}br \perp \frac{1}{9}crr = \frac{1}{9}ss$.

t and $\perp v$ will give $aa \perp bt \perp ctt = vv$.

$-\frac{3}{4}t$ and $\top \frac{3}{4}v$ will give $aa \top \frac{3}{4}bt \perp \frac{9}{16}ctt = \frac{9}{16}vv$.

Now, by exterminating ss from the first and second equations, in order to obtain r, there comes out $\dfrac{2aa}{\perp b} = r$. Whence it is manifest that $\perp b$ is affirmative. Also by exterminating vv from the third

and fourth, to obtain t, there comes out $\dfrac{aa}{3b} = t$. And having writ

$\dfrac{2aa}{b}$ for r in the first, and $\dfrac{aa}{3b}$ for t in the third, there arise $3aa \perp$

$\dfrac{4a^4c}{bb} = ss$, and $\frac{4}{3}aa \perp \dfrac{a^4c}{9bb} = vv$.

Moreover, having let fall BA perpendicular upon CH, BC will be :
AD ($:: 3 : 1$) $::$ BA : AK $::$ CA : DK. Wherefore, since BA is ($=$
AM $-$ AH $= r - t$) $= \dfrac{5aa}{3b}$, AK will be $= \dfrac{5aa}{9b}$, or rather $= .-$

$\dfrac{5aa}{9b}$. Also since CA is ($=$ CH $\perp$ BM $= v \perp s$) $= \sqrt{\left(\dfrac{4aa}{3} \perp \dfrac{a^4c}{9bb} \right)}$

$\perp \sqrt{\left(3aa \perp \dfrac{4a^4c}{bb} \right)}$, it will be DK ($= \frac{1}{3}$CA) $= \sqrt{\left(\dfrac{4aa}{27} \perp \dfrac{a^4c}{81bb} \right)} \perp$

$\sqrt{\left(\frac{1}{3}aa \perp \dfrac{4a^4c}{9bb} \right)}$. Which being respectively written in the equation
$aa + bx \perp cxx = yy$, for AK and DK, or x and y, there comes out
$\dfrac{4aa}{9} \perp \dfrac{25a^4c}{81bb} = \frac{13}{27} \, aa \perp \dfrac{37a^4c}{81bb} \perp 2 \sqrt{\left(\dfrac{4aa}{27} \perp \dfrac{a^4c}{81bb} \right)} \times$

$\sqrt{\left(\dfrac{aa}{3} \perp \dfrac{4a^4c}{9bb} \right)}$. And by reduction $- bb \top 4aac = \perp 2\sqrt{(36b^4}$

$\perp 51aabbc + 4a^4cc)$; and the parts being squared, and again
reduced, there comes out $0 = 143b^4 \perp 196aabbc$, or $\dfrac{-143bb}{196aa} =$

$\perp c$. Whence it is manifest, that $\perp c$ is negative, and consequently
the fictitious equation $aa \perp bx \perp cxx = yy$ will be of this form, $aa +$
$bx - cxx = yy$. And therefore the curve, which it denotes, is an
ellipsis ; whose centre and two axes are thus found.

Making $y = 0$, as happens in the vertex of the figure P and Q, you
will have $aa + bx = cxx$, and having extracted the root $x = \dfrac{b}{2c} \pm$

$\sqrt{\left(\dfrac{bb}{4cc} + \dfrac{aa}{c} \right)} = \left\{ \begin{array}{c} AQ \\ AP \end{array} \right\}$.

And consequently, taking AV $= \dfrac{b}{2c}$, V will be the centre of the

ellipse, and VQ, or VP $\left(\sqrt{\left(\dfrac{bb}{4cc} + \dfrac{aa}{c} \right)} \right)$ the greatest semi-axis.

If, moreover, the value of AV, or $\dfrac{b}{2c}$, be put for x in the equation aa

$+ bx - cxx = yy$. there will come out $aa + \dfrac{bb}{4c} = yy$. Where-

fore $aa + \dfrac{bb}{4c}$ will be $=$ vzq, that is to the square of the least semi-axis. Lastly, in the values of av, vq, and vz already found, writing $\dfrac{143bb}{196aa}$ for c, there comes out $\dfrac{98aa}{143b} =$ av, $\dfrac{112aa\sqrt 3}{143b} =$ vq, and $\dfrac{8a\sqrt 3}{\sqrt{143}} =$ vz.

The other part of the Analysis.

Suppose now the staff ar standing on the point a, and rpq will be the meridional plane, and rpzq the luminous cone whose vertex is r. Let moreover txz be a plane cutting the horizon in vz, and the meridional plane in tvx, which section let be perpendicular to the axis of the world, or of the cone, and the plane txz will be perpendicular to the same axis, and will cut the cone in the periphery of the circle tzx, which will be every where at an equal distance, as rx, rz, rt, from its vertex. Wherefore, if ps be drawn parallel to tx, you will have rs $=$ rp, by reason of the equal quantities rx, rt; and also sx $=$ xq, by reason of the equal quantities pv, vq; whence rx or rz $= \frac12$(rs $+$ rq) is $= \frac12$(rp $+$ rq). Lastly, draw rv, and since vz perpendicularly stands on the plane rpq, (as being the section of the planes perpendicularly standing on the same plane) the triangle rvz will be right-angled at v.

Now making ra $= d$, av $= e$, vp or vq $= f$, and vz $= g$, you will have ap $= f - e$, and rp $= \sqrt{(ff - 2ef + ee + dd)}$. Also aq $= f + e$, and rq $= \sqrt{(ff + 2ef + ee + dd)}$; and consequently

$$\text{rz} \left(= \frac{\text{rp} + \text{rq}}{2} \right) = \frac{\sqrt{(ff - 2ef + ee + dd)} + \sqrt{(ff + 2ef + ee + dd)}}{2}.$$

Whose square $\dfrac{dd + ee + ff}{2} +$

$\frac12 \sqrt{(f^4 - 2eeff + e^4 + 2ddff + 2ddee + d^4)}$ is equal (rvq $+$ vzq $=$ raq $+$ avq $+$ vzq $=$) $dd + ee + gg$. Now having reduced, it is $\sqrt{(f^4 - 2eeff + e^4 + 2ddff + 2ddee + d^4)} = dd + ee - ff + 2gg$, and the parts being squared and reduced into order,

$ddff = ddgg + eegg - ffgg + g^4$, or $\dfrac{ddff}{gg} = dd + ee - ff + gg$. Lastly, 6, $\dfrac{98aa}{143b}$, $\dfrac{112aa\sqrt 3}{143b}$, and $\dfrac{8a\sqrt 3}{\sqrt{143}}$ (the values of ar, av, vq, and vz) being restored for d, e, f, and g, there arises $36 - \dfrac{196a^4}{143b b}$

$+ \dfrac{192aa}{143} = \dfrac{36 \times 14 \times 14aa}{143bb}$, and thence by reduction

$\dfrac{49a^4 + 36 \times 49aa}{48aa + 1287} = bb.$

In the first scheme AMq+MBq is $=$ ABq, that is, $rr + ss = 33 \times 33$; But r was $= \dfrac{2aa}{b}$, and $ss = 3aa - \dfrac{4a^4c}{bb}$, whence $rr = \dfrac{4a^4}{bb}$, and

(substituting $\dfrac{143bb}{196aa}$ for c) $ss = \dfrac{4aa}{49}$. Wherefore $\dfrac{4a^4}{bb} + \dfrac{4aa}{49} = 33$

$\times$ 33, and thence by reduction there again results $\dfrac{4 \times 49a^4}{53361 - 4aa} =$

bb. Putting therefore an equality between the two values of bb, and dividing each part of the equation by 49, you will have $\dfrac{a^4 + 36aa}{48aa + 1287}$

$= \dfrac{4a^4}{53361 - 4aa}$; whose parts being multiplied cross-ways, ordered and divided by 49, there comes out $4a^4 = 981aa + 39204$, whose root aa is $\dfrac{981 + \sqrt{1589625}}{8} = 280, 2254144.$

Above was found $\dfrac{4 \times 49a^4}{53361 - 4aa} = bb$, or $\dfrac{14aa}{\sqrt{(53361 - 4aa)}} =$

b. Whence AV $\left(\dfrac{98aa}{143b}\right)$ is $\dfrac{7\sqrt{(53361 - 4aa)}}{143}$, and VP or VQ

$\left(\dfrac{112aa\sqrt{3}}{143b}\right)$ is $\dfrac{8}{143} \sqrt{(160083 - 12aa)}$. That is, by substituting 280·2254144 for aa, and reducing the terms into decimals, AV $=$ 11·188297 and VP or VQ $= 22 \cdot 147085$; and consequently AP (PV $-$ AV) $= 10 \cdot 958788$, and AQ (AV $+$ VQ) 33·335382.

Lastly, if ½AB or 1 be made radius, ½AQ or 5·555897 will be the tangent of the angle ARQ of 79° 47′ 48″, and ½AP or 1·826465 the tangent of the angle ARP of 61° 17′ 57″, half the sum of which angles 70° 32′ 52″ is the complement of the sun's declination; and the semi-difference 9° 14′ 56″ the complement of the latitude of the place, Therefore the sun's declination was 19° 27′ 10″, and the latitude of the place 80° 45′ 4″ which were to be found.

Questions proposed in 1744, and answered in 1745.

I. QUESTION 244, *by Mr.* John Turner.

Suppose a heavy flexible chain 18 inches long, suspended at its

ends from two pins in the same horizontal line, 12 inches asunder; how far is the lowest point of the chain below the line of the pins: also required a general theorem expressing the area of the space included by the chain and the line joining the pins.

Answered by Mr. N. Farrer.

Let $a =$ the force which keeps the chain in its position at A, $x =$ AC, $y =$ BC, $z =$ AB; draw the tangent BS, and BT $=$ BA perpendicular to the horizon, and TS parallel thereto. Then will the line AB be sustained by three forces; for its gravity acts in the direction BT, it is drawn at A in an horizontal direction by the force a, and it is sustained in B by the tension of the line SB; which three forces are consequently as BT, TS, and BS; as BT $= z$: TS $= a$ :: $\dot{x}$: $\dot{y}$. Take BT $=$ Bb, and draw rn parallel to BS, and Bn perpendicular thereto; then BT is the perpendicular force or weight of the line at B, $\dot{x}$ is the fluxion of the tension at B, whose fluent (x) is the tension or force in the direction nr; but in A where $x = 0$, this tension $= a$, ergo by correction the whole tension drawing in the direction of the curve is $a + x$; then BS $= a + x : z :: \dot{x}$: $\dot{x}$, $\therefore a\dot{x} + x\dot{x} = z\dot{z}$ its fluent is $2ax + xx = zz$, therefore $\dot{y} = \dfrac{a\dot{x}}{\sqrt{(aa + xx)}}$, and $y =$ hyp. log. of $\dfrac{a + x + \sqrt{(2ax + xx)}}{a}$; which equation when $y =$ BQ will give $x =$ AQ $= 6.0314$ the lowest descent; hence $a = \dfrac{zz - xx}{2x} = 3.6992$; the supplemental fluxion $x\dot{y} = \dfrac{ax\dot{z}}{\sqrt{(aa + zz)}}$ gives $az - aa \times$ hyp. log. $\dfrac{z + \sqrt{(aa + zz)}}{a}$, and the area PAQ $= xy - az + aa \times$ hyp. log. $(z + \sqrt{(aa + zz)}) = 25.0916.$

The same answered by Mr. Cockson.

Put DC $= c$, BC $= b$, D$n = x$, $mn = y$, and D$m = z$. Then will $\dot{x} = \dfrac{a\dot{y}}{z}$, $z = \sqrt{(aa + yy)}$; whence, the curve being the catenaria, $y = \sqrt{(aa + zz)} - a$, $\dot{y} = \dfrac{z\dot{z}}{\sqrt{(aa + zz)}}$; this put for $\dot{y}$ in the above value of $\dot{x}$ makes $\dot{x} = \dfrac{a\dot{z}}{\sqrt{(aa + zz)}}$; hence $\dot{y}x$ the flux. of the variable area Dmn is

z 2

$$\frac{a\dot{z}}{\sqrt{(aa + zz)}} \times (\sqrt{(aa + zz)} - a) = a\dot{z} - \frac{aa\dot{z}}{\sqrt{(aa + zz)}}, \text{ whose}$$

fluent is $za - ax =$ the area, and when $z = c$ (x being then $= b$) becomes $(c - b) \times a$ the expression for the area CDE, $\therefore$ the area of the curve is 50·1888, and DB the lowest descent of the chain $=$ 6·0317 inches.

II. QUESTION 245, *by Mr.* John Landen.

I have one hundred pieces of gold ; some of which are pistoles, some guineas, and the rest moidores. Now if a pistole was worth 18s. 6d. a guinea worth 1l. 3s. and a moidore 1l. 10s. my hundred pieces would be worth one hundred pounds. Quere, how many have I of each sort ?

Answered by Mr. Heath.

Let $x =$ the number of moidores, $y =$ guineas ; then $b - y - x$ $=$ the pistoles (putting b for 100 pieces) ; hence $60x + 37b - 37y$ $- 37x + 46y = 4000$ the number of sixpences ; whence $x =$ $\frac{300 - 9y}{23}$ or $y = \frac{300 - 23x}{9}$; from whence x and y are determined in whole numbers 6 and 18, also $b - y - x = 76$.

Remark. This question may also be solved after the manner of several others of the same kind already given in this work. H.

III. QUESTION 246, *by Mr.* Peter Kay.

To find the centre of oscillation of a pendulum, whose bob is composed of two equal and similar parabolical conoids, joined together at their bases ; the thickness of the bob being three inches, the diameter of its greatest circle seven inches, and the distance of its centre from the point of suspension 39·5 inches ?

Solution.

Let A represent the point of suspension. Now here is given A●

$= 39\cdot5 = d$, BO $= 3\cdot5 = m$, and co $= 1\cdot5 = n$:
Let $y =$ SF $=$ SG, $x =$ CS, and put c for the distance of
the centre of oscillation from the point of suspension,

then will c be $= d + \dfrac{\text{the flu. of } y^4 \dot{x}}{2d \times \text{the flu. of } y^2 \dot{x}}$ (by   the

nature of the centre) $= d + \dfrac{\text{the flu. of } m^2 x^2 \dot{x}}{2d \times \text{flu. of } nx\dot{x}} = d$

$+ \dfrac{m^2 x}{3dn} = d + \dfrac{y^2}{3d} =$ (when $y = m$ or $x = n$) d

$+ \dfrac{m^2}{3d} = 39\cdot5 + \dfrac{3\cdot5^2}{3 \times 39\cdot5} = 39\cdot5 + \cdot103376 =$

$39\cdot603376$ the distance required.   H.

Mr. *N. Farrer*'s solution, being erroneous, is omitted.

IV.   QUESTION 247, *by Mr.* J. Botts.

A set of men and women were drinking together, and their reckon-
ing came to just six guineas; towards the discharging of which, each
man agreed to pay a certain sum, and each woman the square root
of the same : Now it was found, if the number of men and women
were mutually changed the one for the other, the reckoning would
have come to half a Portugal piece less, or only to 4l. 10s. Again
it was found, that each man paid as many shillings more than each
woman, as there were women in company. It is required what num-
ber were of each, and what each paid?

Answered by Mr. R. Gibbons.

Put $m =$ the number of men, $n =$ the number of women, and x
$=$ what each woman paid; then, per question, each man paid xx.
Hence $mxx + nx = 126$, $nxx + mx = 90$, and $xx - x = n$. Now
by taking the value of m in the two first equations, we have $\dfrac{126 - nx}{x}$
$= 90 - nxx$. From which and the third equation we get $x^5 - x^4 -$
$x^3 + x^2 = 90x - 126$; whence $x = 3$ the shillings each woman
paid, and $x^2 = 9$ what each man paid; consequently there were 12
men and 6 women.

V.   QUESTION 248, *by Mr.* William Daniel.

In an oblique-angled triangle (EGF) there is given the difference of

the two sides, which compose the oblique angle (ED) $= 2$; the difference of the segments of the base (EB) $= 2\cdot4$; and the oblique angle (EFG) $= 112° 37'$: It is required to find all the other parts of the triangle?

Answered by Mr. Heath.

Put $m = $ s. $\angle$ EFG $= 112° 37'$, [see the following fig.] $x =$ BP $=$ FG, and $n =$ EB the difference of the base's segments; $y =$ DF $=$ FG, and ED $= d$ the difference of the sides containing the oblique angle. Now, per 47 Euc. I, EF² — EF² $=$ PF² $=$ FG² — FG², i. e. $dd + 2dy + yy - nn - 2nx - xx = yy - xx$: whence $\dfrac{dd + 2dy - nn}{2n} = x$ by reduction. Again, EG : s. $\angle$ EFG :: FG : s. $\angle$ FEG, i.e. $2x + n : m :: y :$ $\dfrac{my}{2x + n} = $s.$\angle$ FEG; and rad. : EF :: s. $\angle$ FEG : PF, i. e. $1 : d + y ::$ $\dfrac{my}{2x + n} : (yy - xx)^{\frac{1}{2}}$; whence $yy - xx = \dfrac{mmyy}{4xx + 4x + nn} (dd + 2dy + yy)$, or $(yy - xx)(4xx + 4xn + nn) = mmyy(dd + 2dy' + yy)$. In which if the value of x before found be substituted, an equation will come out in which there will be but one unknown quantity, viz. y.

Answered by Mr. James Rubins.

As EF — GF : EF — FG :: cos. $\dfrac{\angle E + \angle G}{2}$: cos. $\dfrac{\angle G - \angle E}{2}$ (per Thacker's Miscellany, theor. 18, p. 28.) Hence the sides are easily found by plane trigonometry, and EF $= 12\cdot9$, FG $= 10\cdot9$, and EG $= 19\cdot874$.

The same answered by Mr. Arch. Scyth.

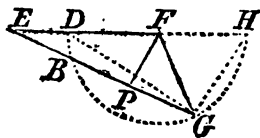In the triangle EFG draw the line DG, then the angle FDG $= \dfrac{\angle G + \angle E}{2}$ and BGD $= \dfrac{\angle G - \angle E}{2}$; but in the triangle EGH, as EG : EH :: s. $\angle$ H : s $\angle$ HGE $(= $ cos. $\dfrac{\angle G - \angle E}{2})$, and in the

$\triangle$ EGD, as EG : ED :: s. EDG : s. EGD $(= $ s. $\dfrac{\angle G - \angle E}{2})$; that is, putting ED $= a$, EB $= b$, FG $= \frac{1}{2}x$, BG $= y$, and c and $d =$ sine and cosine of half the sum of the angles at

the base, $y + b : x + a :: d : d\dfrac{x + a}{y + b} = $ cosine of half the dif-

ference; and $y + b : a :: c : \dfrac{ac}{y + b} = $ s. of half the difference;

$\therefore dd(a + x)^2 + aacc = (y + b)^2$, or $\sqrt{(ddxx + 2add + aa)} = y +$

b: but $(x + a) a = (y + b) b$, $\therefore y + b = \dfrac{xa + aa}{b} = \sqrt{(ddxx}$

$+ 2addx + aa)$, $\therefore xx + 2ax = \dfrac{aabb - a^4}{aa - bbdd}$ and $x = -a \pm$

$\sqrt{\left(\dfrac{aabb - aabbdd}{aa - bbdd}\right)} = \dfrac{abc}{\sqrt{(aa - bb)}} - a = 21{\cdot}96$; and

therefore $\dfrac{abc}{2\sqrt{(aa - bbdd)}} = $ the greater side, $\dfrac{abc}{2\sqrt{(aa - bbdd)}} -$

$a = $ the less, and $\dfrac{ac}{y + b} = {\cdot}056033 = $ s. $3° 12\frac{1}{2}'$, so that the

greater angle is $36° 54'$, and the less $30° 29'$.

Additional Solution.

With the centre F, and radius FG, the shorter side, describe a cir-
cle cutting the base in B, and the longer side EF in D : then ED is $=$
the difference of the sides, and EB $=$ the difference of the segments
of the base. Join DB, GD; then the angle BDG $=$ half BFG, and the
angle BGD $=$ half BFD; therefore EBD $=$ BDG $+$ BGD $=$ half DFG $=$
half the given vertical angle. Whence this construction.

Make EB $=$ the given difference of the segments of the base, and
the angle EBD $=$ half the given vertical angle; from E to BD, apply
ED $=$ the difference of the sides and produce ED. Draw BF to make
the angle DBF, $=$ BDF meeting ED in F; with centre F and radius FD
or FB, describe an arc to cut EB produced in G, and the triangle is
constructed. L.

VI.   QUESTION 249, by Mr. William Brown.

In the latitude $52° 30'$, on the 10th of June (supposing it the
longest apparent day) I asked a mathematical friend, what o'clock
it was? who made me this puzzling answer: Count (says he) the
hours from the visible time of the rising of the sun's centre, and add
their cube root to the square root of the hours to the apparent time
of its setting; and it will give you the hour of the day. Quere,
What o'clock was it?

Answered by Mr. Farrer.

Let $y = $ the hours from sun-rising, n his true time of rising, and

z 4

m the length of the day; then the time to his setting is $m - y$, and per question $\sqrt[3]{y} + \sqrt{(m - y)} = n + y$ the hour of the day: Put $y = x^3$, and we have $x + \sqrt{(m - x^3)} = n + x^3$, $\therefore$ $m - x^3 = n^2 + x^5 + x^2 + 2nx^3 - 2nx - 2x^4$ which in numbers is $x^6 - 2x^4 + 8\cdot4002x^3 + x^2 - 7\cdot4022x = m^2 - n^2 = 2\cdot8997$; hence $x = 1\cdot0919$, and $x^3 = y = 1\cdot3083$, and $n + y = 5\cdot0029 = 5^h\ 0^m\ 10\cdot44^s$ the time required.

<div align="center">VII.   QUESTION 250, <i>by Mr.</i> John Hill.</div>

There is a river, whose stream is divided into two parts; and after running some space the waters are united; between which it has inclosed an island in the form of a geometrical ellipsis, whose transverse diameter is 40 chains (according to Gunter) and conjugate = 30 chains. Upon the transverse diameter is built a farm or cottage house, 132 yards from the centre; and as this piece of land is to be divided by straight hedges from the house to the water, one of them, which should be the shortest that can be made, is to convey the water from the river to fill a cistern by the cellar. It is required to find the shortest distance, and the position it will make with the transverse.

<div align="center"><i>Answered by Mr.</i> Wm. Lax.</div>

Put AC $= t = 20$, DC $= c = 15$, CH $= d = 6$, BH $= a = 26$, AH $= b = 14$; and let PH $= x$.
Then, per property of the curve, we have $tt : cc ::$ $(a + x)(b - x) : cc \times$

$$\frac{ab + bx - ax - xx}{tt} = \text{PW}^2,$$
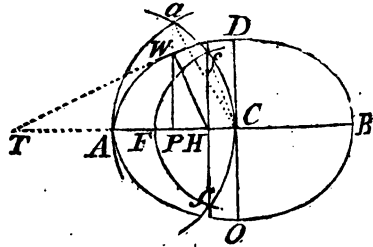

$\therefore cc \times \dfrac{ab + bx - ax - xx}{tt}$

$+ xx = \text{HW}^2$, which by the question is to be a minimum,

$\therefore$ its fluxion $cc \times (\dot{x}b - a\dot{x} - 2\dot{x}x) + 2tt x\dot{x} = 0$; whence $x = \dfrac{cc}{2} \times$

$\dfrac{a - b}{tt - cc}$; or (by writing $2d$ for $a - b$) $x = \dfrac{ccd}{tt - cc} = 169\cdot7$ yards; consequently HW $= 13\cdot36$ chains, making an angle with the transverse of $54° 45'$.

<div align="center"><i>Mr.</i> N. Farrer's <i>answer.</i></div>

Here is given AB $= 40$, OD $= 30$, CH $= 6$ chains: let CB $= m =$

20, $DC = n = 15$, $CH = d = 6$, and $PC = x$; then, per conics, mm

$: nn :: mm - xx : \dfrac{nnmm - nnxx}{mm} = PW^2$, and $\dfrac{n}{m} \sqrt{(mm - xx)}$

$= PW$; $x : m + x :: m - x : \dfrac{mm - xx}{x} =$ subtangent PT; then,

per sim. triangles, $x - d : \dfrac{n}{m} \sqrt{(mm - xx)} :: \dfrac{n}{m} \sqrt{(mm - xx)} :$

$\dfrac{mm - xx}{x}$; therefore $\dfrac{mmx - x^3 - dmm + dxx}{x} = \dfrac{nnmm - nnxx}{mm}$,

therefore $m^4x - m^2x^3 - dm^4 + dmmxx = nnmmx - nnx^3$, that is
$m^2x^3 - n^2x^3 - dmmxx + nnmmx - m^4x + dm^4 = 0$, which divided

by $xx - mm$ gives $mmx - nnx - mmd = 0$, $\therefore x = \dfrac{mmd}{mm - nn} =$

$13 \cdot 714286 = PC$, and $PH = 7 \cdot 714286$, $PW = 10 \cdot 91806$, and $HW =$
$13 \cdot 368406$, making an angle with the transverse of $54° 45' 23''$
nearly.

Additional Solution.

Construction. The point F being the focus, with the centre c and
radius CF describe an arc cutting $fHf \parallel$ OD in f and f; through f, c,
f, describe a circle; and with the centre c and radius CA describe
another intersecting it in a; then awP being drawn $\parallel$ CD will
give the point w in the curve from whence the shortest line WH
must be drawn.

For, having drawn the chords cf, ea, by the nature of the circle
it will be CP : CH :: $(ca^2$ or) CA^2 : $(cf^2$ or) CF^2; hence, by division,
PH (or CP — CH) : CH :: CD^2 (or CA^2 — CF^2) : CF^2; $\therefore$ HW is $\perp$ the
curve by prop. XV. Emerson's Conics, which it ought to be when
it is shortest.

Corollary. Hence, when HW is $\perp$ the curve, CF^2 or CA^2 — CD^2 :
CA^2 :: CH : CP. And from hence also the calculation is very easy;
for CP $= \dfrac{CH \times CA^2}{CA^2 - CD^2} = \dfrac{6 \times 20^2}{20^2 - 15^2} = \dfrac{6 \times 400}{175} = \dfrac{6 \times 16}{7} = \dfrac{96}{7}$

$= 13\frac{5}{7}$. II.

VIII, QUESTION 251, *by Mr. J. Powle.*

To determine the law of the weights, which press each particle of
a perfect flexible line, in such manner as that it shall form a curve,
whose equation is $ax = y^4$?

Answered by Amicus.

Draw the lines EH, FG, HG, and LC will represent the weights upon every particle of any curve: let PM $= \dot{y}$, PH $= \dot{x}$, CF $= a$, AM $= \dot{z}$, and by similar triangles, we shall have $\dot{y} : \dot{x}$

$:: a : LC = \dfrac{a\dot{x}}{\dot{y}}$ whose fluxion is LC $=$

$\dfrac{a\dot{y}\,\ddot{x} - a\dot{x}\,\ddot{y}}{\dot{y}\dot{y}}$, a general expression for all

curves. In the present case we have $a\ddot{x} = \dot{y}^4$, whose fluxion is $\ddot{x} = \dfrac{4\dot{y}^3\ddot{y}}{a}$, which substituting in the above equation

$LC = \dfrac{a\dot{x}}{\dot{y}}$, and taking its fluxion we shall have $\dot{L}C = 12\dot{y}\dot{y}\dot{y}$; and if w expresses the weight that presses the particle M, we shall have $w\dot{z} = 12\dot{y}\dot{y}\dot{y}$, or $\dfrac{w\dot{y}}{a}\sqrt{(aa + 16y^6)} = 12\dot{y}\dot{y}\dot{y}$, or as the cube of the ordinate directly and tangent inversely.   *Q. E. I.*

Answered by Mr. Farrer.

Draw the lines as per figure to the solution of question 244; and let a represent the tension in A, $x =$ AC, $y =$ CB, and $z =$ curve AB. Then we have, by the nature of the curve in the solution to question 214, $z\dot{y} = a\dot{x}$, $2ax + xx = zz$ and $a + x =$ the tension: Now per question $ax = y^4$, its fluxion $a\dot{x} = 4y^3\dot{y}$, $\therefore 4y^3\dot{y} = z\dot{y}$, and $4y^3 = z$ the gravity of the line or required law of the weights pressing every particle of the line; which is as the cubes of the ordinates.

IX.   QUESTION 252, *by Mr.* T. Sandalls.

In an oblique-angled plain triangle, there is given the difference of the sides which include the angle of $112° = 20$, and the perpendicular let fall from the angle on the base $= 60$. Required a theorem to determine the base and sides of the triangle?

Answered by Mr. T. Atkinson.

Given EF — FG $= 20 = 2n$ [see the fig. to question 248] FP $= 60 = m$, s. $\angle$ EFG or 68° $= s$, its cosine $= c$; put $2y =$ EF+FG,

then $y + n = $ EF, and $y - n = $ FG, and per trig. $y + n : 1 :: m :$ $\dfrac{m}{y + n} = $ s. $\angle$ FEP, and $\dfrac{m}{y + n} : y - n :: s : \dfrac{syy - snn}{m} = $ EG ; also $1 : y - n :: c : cy - cn = \frac{1}{2}$FH ; but EG^2=EF^2 + FG^2 + 2EF $\times \frac{1}{2}$FH, i. e, $\dfrac{s^2y^4 - 2s^2n^2y^2 + s^2n^4}{mm} = 2y^2 + 2n^2 + 2cy^2 - 2cn^2$; or $y^4 -$ $\dfrac{2s^2n^2 + 2m^2 + 2cm^2}{ss} \times 2y^2 = \dfrac{2m^2n^2 - 2cm^2n^2 - s^2n^4}{ss}$; or if for the co-efficients of yy and the quantities on the other side of the equation be wrote 2A and B respectively, it will be $y^4 - 2$A$y = $ B ; $\therefore y = \sqrt{(\text{A} + \sqrt{(\text{B} + \text{A}^2)})} = 108\cdot4271$; hence EF $= 118\cdot4271$, FG $= 98\cdot4271$, and EG $= 180\cdot1279$.

The same answered by Mr. W. Kingston.

Let ED $= a = 20$, FP $= p = 60$, s and $c = $ sine and cosine of half the sum of the angles at the base, x and y the sine and cosine of half their difference ; then will $sy + cx$ be the sine of the greater angle at the base, and $sy - cx$ the sine of the less ; also $cy - sx$ and $cy + sx$ their respective cosines ; then as $sy - cx : p :: 1 :$ $\dfrac{p}{sy - cx} = $ EF, and as $sy + cx : p :: 1 : \dfrac{p}{sy + cx} = $ FG ; hence $\dfrac{p}{sy - cx} - \dfrac{p}{sy + cx} = a = \dfrac{2cpx}{ssyy - ccxx}$, $\therefore 2cpx = assyy - accxx :$ But if for yy be put its value $1 - xx$, and for cc, $1 - ss$, it will become $ass - axx = 2cpx$, $\therefore axx + 2cpx = ass$, substitute $2n = \dfrac{2pc}{a}$ and it will be $xx + 2nx = ss$; whence, by compleating the square, and extracting the root, we have $x = \sqrt{(ss + nn)} - n$, and the required sides as above.

Answered by Mr. A. Scyth.

Put $p = 60$, $d = 20$, s and $c = $ the sine and cosine of half the sum of the angles at the base ; $y = $ the base, and $x = $ the sum of the sides ; then will $y : x :: c : \dfrac{cx}{y} = $ the cosine of half the differ-ence of the angles at the base, and $y : d :: s : \dfrac{sd}{y} = $ its sine, $\therefore ccxx + ssdd = yy$, and $xx = \dfrac{yy - ssdd}{cc}$ but $\dfrac{csx + csd}{y} = $ the sine of the greater, and $\dfrac{cxs - csd}{y} = $ the sine of the less, therefore

$$\frac{csx + csd}{y} : p :: 1 : \frac{py}{cs \times (x + d)} = \text{the lesser side, } \therefore \frac{py}{cs\,(x - d)}$$

$=$ the greater, and their sum $\dfrac{2pyx}{sc\,(xx - dd)} = x, \therefore xx = \dfrac{2py}{sc} + dd =$

$\dfrac{yy - ssdd}{cc}$, or $yy - \dfrac{2pcy}{s} = ccdd + ssdd = dd$; which reduced gives

$$y = \frac{pc \pm \sqrt{(ppcc + ssdd)}}{s} = 180 \cdot 122; \text{ whence } \frac{x \pm d}{2} =$$

$$\frac{\sqrt{\left(dd + \dfrac{2p}{ssc} \times (pc + \sqrt{(ppcc + ssdd)}) \right)}}{2} \pm \frac{d}{2} = 118 \cdot 42 \text{ the}$$

greater side, or 98·42 the less.

Additional Solution.

Let ACB be the triangle required, draw CP perpendicular to AB
and take $ca = ca$ and make the triangle cGD
equal in all respects to CAP, viz. the angle
DCG $=$ ACP and the angle CDG $=$ APC $=$ a
right angle; then the angle PCD is $=$ the given
angle ACB, CD $=$ the given perpendicular CP,
and GB $=$ the difference of the sides; and the
line DH being perpendicular to CD, it is given
by position with respect to AB; whence we
have only to draw the line CB through the
given point c, which is equidistant from the lines DH, AB, so that the
part GB intercepted by those lines may be of a given length, which is
one of the problems of Inclinations, and which is constructed in the
additional solution to question 199, and in several other places.
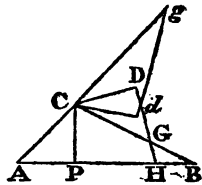
If the sum of the sides were given instead of the difference, then
if cg be taken equal to CB, on AC produced, and the triangle cgd be
made equal in all respects to BCP, namely, the angle cgd $=$ CBP and
the angle at d a right angle, then the angle PCd will be the supple-
ment of the given angle ACB, and the line dg will be given by position
as before, and we shall have to draw Ag through the given point c,
so as to be equal to the given sum of the sides.

When the given angle is right, as in quest. 22, the lines CD and cd
coincide, and are parallel to AB, and consequently DH and dg are
then perpendicular to that line. L.

X.   QUESTION 253, by Mr. J. Powle.

Granting the resistance, as the square of the celerity; in what law
of density will a heavy body moving describe a curve, whose equa-
tion is $ax = y^s$?

Answered by Mr. N. Farrer.

Let Arc represent the curve described; draw the lines as in the figure, and let the velocity at r in the direction $rc =$ v, AH $= x$, $rn = \dot{x}$, Hr $= y$, $mn = \dot{y}$, $rm = \dot{z}$, the required density as D, c the celerity, and law of resis-

tance as ac^n; then $\dot{y} : v :: \dot{x} : \dfrac{v\dot{x}}{y} \doteq$ the velocity
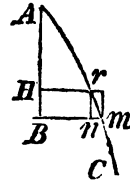
in the direction nr, its fluxion is $\dfrac{v\ddot{x} + \dot{v}\dot{x}}{y} =$ the in-

crease of velocity during the time of describing rm; $\dot{y} : \dot{x} :: \dot{v} : \dfrac{\dot{v}\dot{x}}{\dot{y}}$

the part arising from the resistance of the medium; therefore $\dfrac{v\ddot{x}}{y} =$ the part arising from the force of gravity. The resistance is to the

force of gravity as $-\dfrac{v\ddot{x}}{\dot{y}}$ to $\dfrac{v\ddot{x}}{\dot{y}}$, or as $-\dfrac{v\ddot{x}}{\ddot{v}\dot{x}}$ to 1; but $\dfrac{v\ddot{x}}{\dot{y}}$, the ve-

locity arising from gravity, being proportional to the time $\dfrac{\dot{y}}{v}$ of de-

scribing rm, may be expressed thereby; hence $\dfrac{v\ddot{x}}{\dot{y}} = \dfrac{\dot{y}}{v}$ or $vv\ddot{x}$

$= \dot{y}\dot{y}$, in fluxions $2\dot{v}v\dot{x} + vv\ddot{x} = 0$, or $-\dfrac{\dot{v}}{v} = \dfrac{\ddot{x}}{2\dot{x}}$, which substi-

tuted in the foregoing proportion $-\dfrac{\ddot{v}z}{\dot{v}x} : 1$, gives $\dfrac{\ddot{z}x}{2\dot{x}^2} : 1$ the ratio of the resistance to the gravity.

Again, since the absolute velocity is $\dfrac{v\dot{z}}{y}$ the resistance by supposi-

tion will be $a \times \left(\dfrac{v\dot{z}}{y}\right)^n$; hence D as $\dfrac{\ddot{x}}{a\dot{z}^{n-1} \times \ddot{x}^{\frac{4-n}{2}}}$ which

when $n = 2$, or the resistance as the square of the velocity will be

$\dfrac{\ddot{x}}{\dot{x}}$: But, per question, the equation of the curve is $ax = y^3$, $\therefore a\dot{x} = 3y^2\dot{y}$,

$\ddot{x} = \dfrac{6yy^2}{a}$, and $\therefore \dddot{x} = \dfrac{6\dot{y}^3}{a}$, $\therefore$ the density $\dfrac{\ddot{x}}{\dddot{x}}$ will be as

$\dfrac{1}{\frac{\dot{y}}{a}\sqrt{(9y^4 + a^2)}}$, or as the tangents reciprocally.

Answered by the Proposer.

Because the resistance is as the square of the celerity, $\dfrac{\ddot{x}}{\dot{x}\sqrt{(\dot{x}^2 + \dot{y}^2)}}$

(Simpson's Essays, p. 60) will express the required density : But by

the equation of the curve $(ax = y^3)$ we have $\dot{x} = \dfrac{3y^2\dot{y}}{a}$, $\ddot{x} = \dfrac{6y\dot{y}^2}{a}$,

$\therefore \dddot{x} = \dfrac{6\dot{y}^3}{a}$, and $\sqrt{(\dot{x}^2 + \dot{y}^2)} = \dfrac{\dot{y}}{a}\sqrt{(a^2 + 9y^4)}$; therefore, by substi-

tution, the required density becomes $\dfrac{6\dot{y}^3}{\dfrac{6y\dot{y}^2}{a}\sqrt{(a^2 + 9y^4)}} =$

$\dfrac{a}{y\sqrt{(a^2 + 9y^4)}}$; or by a farther reduction $= \dfrac{1}{\sqrt{(yy + 9xx)}}$; i. e;

it is always as the tangent of the distance from the vertex to unity.

xı.   ǫᴜᴇsᴛɪᴏɴ 254, *by* Diophantus.

Since the doctrine of triangles has an unbounded use and application in most parts of the mathematics, and the similarity of them generally had recourse to; let it be required to find eight right-angled plain triangles, whose hypothenuses are all equal; and shew a general method for determining the same?

Answered.

Let $bb =$ the square of the common hypothenuse, and xx the square of one leg, then will $bb - xx$ be the square of the other leg. Suppose $b - nx =$ that leg whose square is $bb - xx$, then will $bb - xx$ $= bb - 2bnx + nnxx$, which equation reduced gives $x = \dfrac{2bn}{1 + nn}$ $=$ to one leg, and if instead of x in $b - nx$ we take its value last

found, we shall have $\dfrac{1 \, \infty \, nn}{1 + nn} =$ the other leg; since n may be as-
sumed at pleasure, not only eight, but any number of right-angled
plain triangles whatsoever may be found; which will also appear by
making the common hypothenuse the diameter of a circle.

XII. QUESTION 255.

Being at so large a distance from the dial-plate of a great clock,
that I could not distinguish the figures; but as the hour and minute
hands were very bright and glaring, I could perceive, that the mi-
nute hand pointed upwards to the right hand, at the same time the
hour index pointed downwards to the left, so as both were in a right
line, or diametrically opposite, and in such position, as that the ele-
vation (I guessed) was some few degrees more than 50 above the
horizon. Quere, The hour and minute of the day.

Answered by Mr. J. Landen.

By considering the question, I find that the first time the hands
are diametrically opposite after 6 o'clock, is the time of the day
sought. Let x hours be the time from 6: now the minute hand go-
ing round once in an hour, the rounds it will be carried in the time
x, may be represented by x; and the hour hand going round once in
12 hours, the part of the circumference it will be carried in the same
time by $\frac{1}{12}x$. It is evident that the index whose motion is swiftest,
will outgo the slowest, one circumference in x hours; whence $x -$
$\frac{1}{12}x = 1$, and $x = \frac{12}{11}$; therefore the time required is $5^m \, 27^s$ past 7.

Answered by Mr. James Terey.

By the data it appears to be somewhat past 7. Let $x =$ the num-
ber of minutes past 7; then will $\frac{1}{12}x$ be the distance of the hour hand
from 7, and $25 - \frac{1}{12}x$ the distance of the hour hand from 12; hence
$25 + x - \frac{1}{12}x = 30'$; $\therefore x = \frac{60}{11} = 5\frac{5}{11}$ past 7, the time required.

THE PRIZE QUESTION, by Mr. J. May, jun. of Amsterdam.

An architect, or master carpenter, in Holland, had (from that
slender knowledge which usually attends mechanics) conceit enough
to fancy, he could find the dimensions of any piece of timber in a

building, of which a design should be given : A burgo-master of the
city of Amsterdam, intending to build a handsome house fronting
the street, where his length was limited, because he would save the
charges of double roof and gutter, and at the same time put his best
side outward, gives the said architect these dimensions, viz. That
the building should be forty feet wide, and the front wall twelve
feet higher than the back wall : Also, because too much of a large
roof should not appear in view of the street, he will have the length
of the rafters, from wall to ridge, on the back side of his house,
just 37 feet ; but the rafters on the front side to be of such a length,
as may form the pitch, or steepness of the roof, the same on each side.
The owner being frugal (not to say wise) orders the builder to sit
down and count up the cost : But although he was skilful in numbers,
and pretty well versed in some parts of geometry, yet he found the first
would be so much adfected, and the latter only an approximation,
that he was not able to know how high the roof would rise, nor the
length of the rafters in front, and therefore was incapable of computing
the timber and roofing. The burgo-master surprized, probably
thinking so famed an architect must be little less than a conjuror
(when himself was none) resolves not to have his house begun until
he can have the measures exact, and leaves him bare-breach'd, riding
on the strange roof, although he is furnished with mathematical
instruments, to describe curves and conic sections organically.
But having heard of such things being effected by geometrical con-
struction, he has, through the mediation of a friend, applied to the
artists of Great Britain ; and thinking the author of the Ladies' Diary
deals in quibbles and quaint questions, hopes to see both methods in
the next year's production.

Answered by Mr. Ja. Terey.

Put BC, the height of the front wall above the back, $= 12 = d$,
$BD = 40 = a$ the breadth of the house,
$GD = GF = 37 = b$, and CG the re-
quired length of the front rafters $=$
x ; then $EF = \sqrt{(xx + 2bx + bb - aa)}$, and $AC : BC :: CF : FE$, i. e. $b - x$
$: d :: b + x : \sqrt{(xx + 2bx + bb - aa)}$, therefore $db + dx = (b - x)$
$\sqrt{(xx + 2bx + bb - aa)}$; which
equation, ordered,

$$x^4 - \begin{matrix} -2bb \\ aaxx \\ -dd \end{matrix} + \begin{matrix} +2baa \\ -2ddb \end{matrix} x = \begin{matrix} ddbb \\ bbaa. \\ -b^4 \end{matrix} \text{ In}$$

numbers, $x^4 - 4482xx + 107744x = 513375$; whence $x = 6·51224$ &c. or $x = 20·21768$ &c. either of which determines the length of the front rafters.

Geometrically thus: Let AG $=$ 37 always pass through the point c, the point A sliding along the line AB; then will G the point at the other end of the line AG describe the curve CGGL. On D as a centre with the radius $=$ 37, describe a circle, and it will cut the said curve in the points G, G, which determines the length and position of the front rafters, as is evident by inspection.

Or let AF $=$ 2AG be moved as before, and it will cut the perpendicular DF in the points F and F, whence CG, &c. is known as before.

N. B. If AF, CB, CI, and IM be called *a*, *b*, *x*, and *y* respectively; then will $\dfrac{xxaa}{xx + 2bx + bb} - xx = yy$ express the nature of all such curves.

The same answered by Mr. John Corbett.

Put $x =$ sine of the angle AFH $=$ GCB, whose cosine is $\sqrt{(1 - xx)}$, radius 1; BC $=$ 37 $= b$, HA $=$ 12 $= c$, and HC $=$ 40 $= d$: Then as $x : c :: 1 :$ $c \div x =$ FA, and as $1 : b ::$ $\sqrt{(1-xx)}$ $: b\sqrt{(1 - xx)} =$ GC, for the triangles
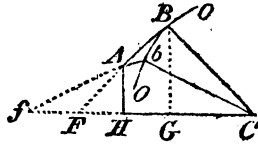
BGC and AHF are alike. As $1 : \dfrac{c}{x}$ (=

FA) :: $\sqrt{(1 - xx)} : \dfrac{c}{x}\sqrt{(1-xx)} =$ FH; consequently $\dfrac{c}{x}\sqrt{(1-xx)}$ $+ d = 92b\sqrt{(1 - xx)}$; which reduced, gives $x^4 - ·324324x^3 - ·681519x^2 + ·324324x = ·0262965$. Two of the roots of which equation are $x = ·71507$ and $x = ·39403$, which in the tables answer to 45° 39′ and 23° 12′; and thence the side AB $=$ 20·22 and AB $=$ 6·54 are found, either of which lengths will answer the conditions of the question.

To effect this geometrically, Extend the line HC $=$ 40 to any distance $\mathfrak{r}f$, at one end of which H erect the perpendicular HA $=$ 12; from the other end C with the extent of 37 describe the arc oo; then lay a ruler upon the point A, and move it about till it cuts the arc oo and the line fH at a distance $=$ BC; so shall BA or bA be the length of the roof sought $=$ 20·22 or 6·54 as above.

Answered by the Proposer.

On BD let fall a perpendicular from A, being supposed already

found. Prolong BD and AC to
K. Put BI or IK $= x$, and AI
$= y$; then is BK $= 2x$, from
which subtract BD $= a$, and
there will remain DK $= 2x - a$.
Now DK $(2x - a)$: DC (b) ::
IK (x) : AI (y), $\therefore 2xy - ay =$
bx, an equation belonging to an
hyperbola. Put x infinite, then
is $y = \frac{1}{2}b$; and by putting y
infinite, $x = \frac{1}{2}a$. Hence, if
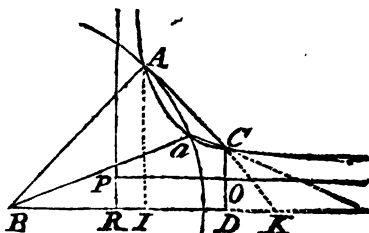
BD and DC are bisected, and the lines PO, PR drawn respectively pa-
rallel thereto, they will be two asymptotes to the said hyperbola.
To find a point in the curve, put $x = a$ in the hyperbolic equation,
then is $y = b$. Consequently c will be a point in the curve, through
which draw the hyperbola Aac.

Per 47 Euc. I. BI^2 + AI^2 = AB^2 ($= cc$), or $xx = cc - yy$ (which
is an equation belonging to a circle), therefore $x = \sqrt{(cc - yy)}$.
Put the surd $= 0$, then $x = 0$, whence the centre will be in B;
whence also $y = c$, or $y = - c$, which determines the radius.
Therefore, if on the centre B, with the radius AB, be described the
arc Aa, which cuts the above constructed hyperbola in the points A,
a, and the lines AB and AC, or aB and ac, be drawn, they will be
the sides of the roof required.

This question admits likewise of a
very easy and elegant solution, by
the help of another curve, and is thus
performed: BD and DC being drawn
as before, and likewise on the centre B
the arch Aa; lengthen the line DC,
drawing CV parallel to BD, and sup-
pose an infinite number of radii drawn
from the centre B, on which make FG,
&c. $=$ FE, &c. drawing through the
points G, G, G, &c. the curve, whose
equation is
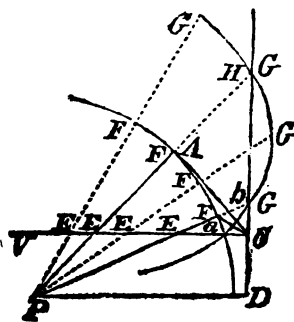
$$+ xx$$
$$y^3 + 2yb - 4aay^2 + 2xxby + xxbb = 0,$$ and it will cut the line
$$+ bb$$

DC in H and h. Lastly, draw BH and Bh, which cut the above circle
in A and a, as before.

Questions proposed in 1745, and answered in 1746.

I.   QUESTION 256, *by Mr.* J. Turner.

The inside dimensions of an apothecary's mortar were found to be as follows : top diameter 12, bottom diameter 6, and depth 9 inches. The internal superficies of the mortar was formed by the revolution, of a conic parabola about an axis parallel to the principal diameter of the curve, the vertex being in the edge of the top diameter. Required the content in wine gallons?

Answered by Mr. J. Landen.

Let ABCD represent the mortar, generated by the revolution of the parabolic curve BD, about the axis EF, parallel to BH.   Call EB, b ; BH, x ; GH, y ; and put $3.1416 = p$.   Then will $p\dot{x}\,(bb - 2by + yy)$ be the fluxion of the required solidity, which, by putting for y and yy, their values $x^{\frac{1}{2}}$ and x (found by the equation of the curve) will be $p\dot{x}\,(bb + x - 2b\sqrt{x})$ and its fluent $pbbx + \frac{1}{2}pxx - \frac{4}{3}pbx\sqrt{x}$; which, when $x = 9 = $ EF, will be 466.5276 inches $= 2.0108$, &c. wine gallons?
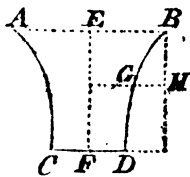
II.   QUESTION 257, *by Mr.* T. Cowper.

On the 12th of February, 1744, in lat. 52° 20′, at 53 minutes past 5, apparent time, the altitude of the nucleus of a comet was observed to be 5¼°, and its distance from the planet Venus 56⅔°. Required the comet's place, allowing for refraction?

Answered by Bironnos.

The place of Venus at the given time is ♑ 20° 50′ 49″ her latitude 1° 22′ 7″ N. declination 20° 27′ south, and is 44° 29′ 45″ short of the meridian: Hence there is known Z⊙, the comet's true zenith distance $= 84° 48′ 35″$, ZP, the co-latitude of place, $= 37° 40′$, ∠ZPO $= 44° 29′ 45″$, OP the distance of Venus from the pole $= 110° 27′$, and O⊙ the comet's distance from Venus $= 56° 36′$.   In △ZPO is known ZP, PO, and ∠P: Find the ∠ZOP $= 25° 35′ 49″$ and ZO $= 82° 25′ 41″$.   In the △ZO⊙, are known the three sides : Find the ∠ZO⊙ $= 88° 44′ 54″$: From which
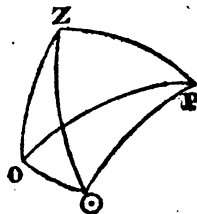
take the $\angle$ zop, and there remains the $\angle$ po$\odot$ = 63° 9′ 5″. Then, in the $\triangle$ p$\odot$, is known po, $\odot$o, and $\angle$po$\odot$: Find the $\angle$op$\odot$ = 48° 59′ 54″, and po$\odot$ = 80° 44′ 20″. Therefore the comet's declination is 9° 15′ 40″ N. and right ascen. 341° 19′ 34″. Hence the comet's longitude is $\mathcal{X}$ 16° 25′ 26″, and latitude 15° 54′ 54″ N.

III. QUESTION 258, *by Mr.* J. Powle.

Given $1 + 2 + 3 + 4$, &c. continued to x terms; to find $1 + 4 + 9 + 16$, &c. the sum of the squares of those numbers.

Answered by Mr. J. Powle.

Put S = $1 + 4 + 9 + 16$, &c. s = the next succeeding term in the series to the *last*, or the increment of s when the number of terms are $x + 1$. Then, because the first term and common difference are each unity, $x + 1 = \sqrt{s}$, which in the series $xx\overset{''}{x}x$, &c. answers to x, $\therefore$ s = xx, or $xx + x$. Whence s = $\frac{1}{3}xxx + \frac{1}{2}xx$ = (by substitution) $\frac{1}{3}(x + 1)(x - 1) x + \frac{1}{2}(x + 1) x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x.$

Answered by Bironnos.

Let Ax^{n+1} + Bx^n + cx^{n-1} + Dx^{n-2} &c. .... + L = 1^n + $2^n + 3^n$ &c. to x^n; then will A$(x + 1)^{n-1}$ + B$(x + 1)^n$ + c$(x + 1)^{n-1}$ &c..... + L = 1^n + 2^n + 3^n &c. to $xn + (x+n)^n$; from which subtract the former, and there remains A $\times$ $((x + 1)^{n+1} - x^{n+1})$ + B $\times$ $((x + 1)^n - x^n)$ + c $\times$ $((x + 1)^{n-1} - x^{n-1})$ &c. = $(x + 1)^n$. By expanding the several powers of $x + 1$ we get A = $\frac{1}{n + 1}$, B = $\frac{1}{2}$, c = $\frac{n}{3.4}$, D = 0, E = $\frac{n \times n - 1 \times n - 2}{2.3.4.5.6}$.

Consequently 1^n + 2^n + 3^n &c. = $\frac{x^{n+1}}{n + 1}$ + $\frac{x^n}{2}$ + $\frac{nx^{n-1}}{3.4}$ − $\frac{n \times n - 1 \times n - 2}{2.3.4.5.6} x^{n-3}$ &c. Which when $n = 2$ will become $\frac{2x^3 + 3x^2 + x}{6}$, the theorem required.

IV. QUESTION 259, *by Mr.* Landen.

A cannon ball projected from the ground, in a direction making

an angle of 14° 29′ with the horizon, fell at the feet of a person some distance off the very moment he heard the report of the piece. Quere, how far he was from the place of projection?

Answered by Mr. N. Farrer.

Let ASD represent the path of the projectile, thrown from A, with a velocity that will carry it in a perpendicular ascent to K; or in the direction AL, to L; and let this celerity carry it through the distance d in the time u, and the distance run through by a falling body in that time $= w$. Put any distance AB $= y$, s and e the sine and cosine of the angle of direction LAD, then will $\dfrac{usy}{de}$ be the time in which
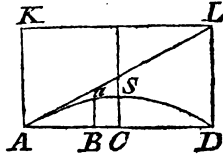
the projectile runs through the curve Aa, and $w \left(\dfrac{usy}{de}\right)^{2}$ the distance descended by a heavy body in that time; ∴ Ba = $\dfrac{seddy - wuussyy}{ddee}$; and when this $=$ cs, then $y = \dfrac{sedd}{w}$, and the time of description $= \dfrac{sdes}{w}$. Put 1142 the feet sound moves in 1 second $= q$; then $q : 1 :: \dfrac{sedd}{w} : \dfrac{sdu}{w}$, therefore $d = \dfrac{qu}{c}$; consequently $\dfrac{sqquu}{ew} =$ the required distance. Now let $n = 1''$, then $w=$ $16\frac{1}{12}$ feet; hence AD $= 20895{\cdot}106$ feet $= 3$ m. 7 f. 21 p. and 2 yards nearly, and the time of description $\dfrac{sdu}{w} = 18{\cdot}29$ seconds.

Mr. Tho. Cowper's answer.

Let AL be the line of direction, and AD the distance of the person from the place of projection. Put $t =$ tangent of the ∠DAL; $d = 16\frac{1}{12}$ feet, the perpendicular descent of heavy bodies in 1 second; $b = 1142$ feet, the velocity of sound in the same time; and $x =$ the time the ball was in motion. Then $bx =$ AD, and $dxx =$ DL; but as $1 : t :: bx : tbx =$ DL, ∴ $x = tb \div d$; hence $tbb \div d =$ AD $= 20945{\cdot}59$ feet, the distance sought.

v. QUESTION 260, by Mr. N. Farrer.

In an oblique-angled triangular grove, one of whose sides is 20 chains, and the angle opposite thereto 78° 45′, if a perpendicular be let fall from each angle to its opposite side, they intersect at a

fountain within the grove, whose nearest distance to the given side
is 8·19 chains : Quere, its distance from each of the other sides, by
a simple equation?

Answered by Mr. Heath.

It is evident by the data that the triangle must be isosceles. In
the △ABC, the ∠A = ∠C, or the ∠B must
be the given angle; and either way the
∠ArB = ∠crB, or ∠Arc is given = 101°
15′ opposite to the side given = 20 chains,
on which describing a segment of a circle to
contain it, and drawing the parallel dis-
tance 8·19 (or rather 8·2 as it should be)
chains = cr, and it will be found a maxi-
mum (as is also proved by trigonometry),
therefore the data are rc, AC = cc, and ∠B, ∠A = ∠C, and ∠BAb;
whence follow by trigonometry Ar = cr = 12·925, ar = rb =
2·5215 (AB = BC = 15.64) chains required.
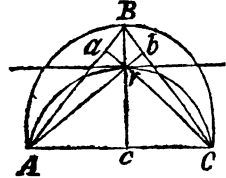
Additional Solution.

The meaning of this prob. is thus : In a triangle we have given
the base (AC), the vertical angle (ABC), and the distance (cr) of the
base from the common point of intersection of three lines drawn from
the three angles perpendicular to the opposite sides ; to determine the
triangle.

We are not at liberty to suppose the triangle isosceles, for that
would be introducing a condition too much into the problem ; and
whether it be isosceles or of any other form, can only appear from
the construction or calculation.

Construction.   On the given base describe two segments of circles,
the one ABC to contain the given vertical angle, and the other Arc to
contain its supplement; parallel to AC, and at the given distance of
the point from it, draw a line to cut or touch Arc in r; then through
r draw crb ⊥ AC, and B will be the vertex of ABC the triangle re-
quired.

For, through r drawing Arb and crc, since by the construction AC
is the given base, ABC the given vertical angle, and rc the given dis-
tance, we have only to prove that the angles at a and b are right
angles.   Now by the construction it appears that Arc, ABC are seg-
ments of equal circles, and that the two segments together make up
a whole circle; then the ∠ABr or aBr = ∠Acr as standing on equal
segments Ar, and the opposite angles rab, crc are also equal; ∴ the
third angles are equal, that is ∠a = ∠c = a right angle.   In the
like manner b is proved to be a right angle.

Scholium.   Another method of construction might be by first
finding the point r as before, through which draw the indefinite lines

Ar*b*, cr*a*, and perpendicular to them the lines c*b*B, A*a*B.——And then we should have to prove that these last two lines and cr, produced, meet in the same point B, and that ABC is = the given angle.

Corollary 1. The equal opposite angles A*r*a, cr*b*, being the supplements of the ∠A*r*c, are each = the ∠ABC, which is also the supplement of A*r*c by the construction.

Corollary 2. Hence also the ∠ BA*r* = BC*r*, and the four triangles BA*b*, *a*A*r*, BC*a*, bcr are all similar.

The Method of Calculation will be, first to calculate the angles *r*AC, *r*CA, of the △ A*r*c, by prob. V. Simpson's Algebra, and then the segments A*c*, c*c*; then to each of these two angles adding the ∠BA*b* or BC*a* = the comp. of ∠ABC, there will be had the ∠BAC and BCA; from which, and the segments A*c*, c*c*, the two hypothenuses AB, BC are easily got, and will come out 15·386 and 16·127, the angles at A and c being 52° 16′ and 48° 59′. H.

VI. QUESTION 261, *by Mr.* C. Cockson.

There are two ponds of water of the same quality and depth, under the same meridian, one in the lat. of $a - 16b$* north, and the other $8b = 5125$ miles due north from it. In the year 1743-4, the 16th of January, at three of the clock in the morning, the thickness of the ice in the southermost was 6 inches. Quere the latitude, and thickness of the ice of the nothermost pond at the same time?

$$* \frac{\sqrt{aaa} + \sqrt[4]{aaa} - \sqrt[3]{aa}}{b} = a.$$

Answered by Mr. N. Farrer.

Given $\dfrac{\sqrt{aaa} + \sqrt[4]{aaa} - \sqrt[3]{aa}}{b} = a.$ In which write $y^{12} = a$, and it is $y^{10} + y - 640·625y^4 - 1 = 0.$ whence $y = 2·936056$, and $y^{12} = a = 410367$; from which subtract $16b = 10250$ and there remains 400117 miles $= 6668°$ 37′, from which subtract 360° × 18 = 6480° 0′, and there remains 188° 37′. Hence 188° 37′ — 180° 0′ $= 8°$ 37′ S. the latitude of the southermost pond; which taken from 85° 25′, leaves 76° 48′ N. the latitude of the northermost. In lat. 8° 37′ S. sun's depression $= 34°$ 35′, continuance under the horizon $8^h 48^m$. In lat. 76° 48′ N. sun's depression $= 27°$ 48′; continuance under the horizon 91 days. And, supposing the intensity of cold as the sines of the sun's depression multiplied by the time of his continuance under the horizon, and the solidity or thickness of ice in the subtriplicate ratio t . . . of,w have $\sqrt[3]{·208}$: $\sqrt[3]{·2·441}$ :: (·5926 : 3·488 ::) 6 inches : 3·31 inches, the thickness of the ice of the northermost pond.

A A 4

VII. QUESTION 262, *by Mr.* Powle.

To determine the asymptotes of a curve whose equation is $x^5 - y^5$ $+ axy = 0$,

Solution.

In the given equation $x^5 - y^5 + axy = 0$, supposing x to be infinite, the term axy will vanish in comparison of x^5 or y^5, and then $x^5 - y^5 = 0$, and $x = y$; that is, at an infinite distance the abscissa is $=$ the ordinate, and therefore the asymptote, or tangent at the infinite distance, must make an angle of 45° with the abscissa. Again the given equation in fluxions

gives $\dot{x} = \dfrac{5y^4 - ax}{5x^4 + ay}\dot{y}$, hence the subtangent

BT or $\dfrac{y\dot{x}}{\dot{y}}$ is $= \dfrac{5y^5 - axy}{5x^4 + ay} =$ (by expunging y^5) $\dfrac{5x^5 + 4axy}{5x^4 + ay}$, and

consequently $vr = BT - x = \dfrac{3axy}{5x^4 + ay} =$ (when $x = y =$ infinite)

$\dfrac{3a}{5x^2} = 0$, and therefore the asymptote passes through the vertex v and

makes an angle of 45° with vB. And the form of the curve is as represented in the figure, where va is the asymptote, H.
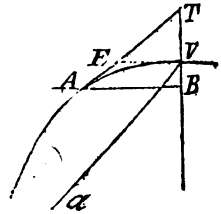
Another Solution by Mr. Landen, *taken from the Appendix to Dr.* Hutton's *Edition of the Diaries.*

Of $x^5 - y^5 + axy = 0$, the equation of the curve, take the fluxions; and for $\dot{x}$ and $\dot{y}$ put the invariable quantities m and n respectively : repeating the operation till you get an equation consisting only of invariable quantities. By so doing you will get

$$5mx^4 - 5ny^4 + amy + anx = 0,$$
$$10m^2x^3 - 10n^2y^3 + amn = 0,$$
$$m^3x^2 - n^3y^2 = 0,$$
$$m^4x - n^4y = 0,$$
$$m^5 - n^5 = 0:$$

each of which equations, shewing the relation of x and y, will (as appears by chap. IX, of my *Resid. Anal.*) express a line which will be an asymptote to the curve expressed by the given equation, when $\dfrac{m}{n}$ is determined by the final equation $m^5 - \bullet^5 = 0$,

Now by that final equation $\dfrac{m}{n}$ is $= 1$; therefore each of the equations $5x^4 - 5y^4 + ay + ax = 0$, $10x^3 - 10y^3 + a = 0$, $xx - yy = 0$, $x - y = 0$, is an asymptotic equation with respect to the proposed curve.

The 4th asymptotic equation corresponds to a right line, passing through the point where x begins, making an angle of 45° with the base on which x is measured. The 3d asymptotic equation corresponds to two right lines; one of which is the same as that expressed by the 4th asymptotic equation; and the other, expressed by $x + y = 0$, is (by what is said chap. X, of my *Resid. Anal.*) a diameter of the proposed curve; which diameter cuts the asymptote at right angles. The 2d asymptotic equation corresponds to a line of the 3d order (which is Sir Isaac Newton's 45th species) having the same asymptote as the proposed curve. The 1st asymptotic equation likewise appertains to a line of the the 3d order (of the species just now mentioned), having the same asymptote as the proposed curve, and to the right line which I have observed in the diameter of that curve. And the form of the proposed curve, with its rectilineal asymptote and diameter, is (by chap. VIII, IX, and X of my *Resid. Anal.*) as in the marginal figure; x being measured on AB (a tangent at the node A) to which the ordinate BC (y is perpendicular; AD the asymptote; AE the diameter.
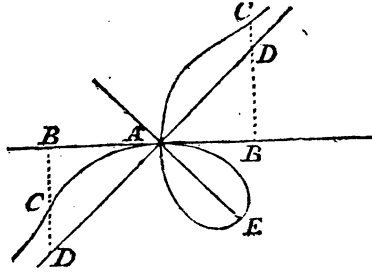
Mr. N. Farrer's solution being very erroneous is omitted.

VIII. QUESTION 263.

Let AFK be the conchoid of Nicomedes, and BC the asymptote whose length is 60, and P the pole'; a line drawn from P perpendicular to the asymptote, to the curve at A, is 40; also from the pole to the asymptote is 20. Required the length of the curve line AFK, with the analytical investigation?
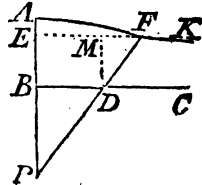
Answered by Mr. Heath.

To rectify the conchoid of Nicomedes generally. Let $b = $ PB, $a = $ BA ($ = b = 20$), $y = $ EF any ordinate, and $x = $ MD $= $ BE.

Then because DP is always equal to BA, $EF = \dfrac{b + x}{x} \sqrt{(aa - xx)}$

and $AE = a - x$, its corresponding abscissa. In fluxions — $\dot{x}$ ×

$\dfrac{x^3 + baa}{xx \sqrt{(aa - xx)}} = \dot{y}$, whence $\sqrt{(\dot{y}\dot{y} + \dot{x}\dot{x})} = \dfrac{a\dot{x}}{xx} \times$

$\dfrac{\sqrt{(bb + aa + 2bx^3 + x^4)}}{\sqrt{(aa - xx)}} = \dfrac{a\dot{x}}{xx} \times : ba + \dfrac{x^3}{a} + \dfrac{x^4}{2ba} - \dfrac{x^6}{2ba^3}$ &c:

$\times : \dfrac{1}{a} + \dfrac{x^2}{2a^3} + \dfrac{3x^4}{8a^5}$ &c. the fluent of which terms collected is

$\dfrac{bx}{2a} + \dfrac{x^2}{2a} + \dfrac{x^3}{60a} + \dfrac{bx^3}{8a^3} + \dfrac{x^4}{8a^3} + \dfrac{x^6}{16a^3}$ &c. $- \dfrac{ba}{x} - \dfrac{x^5}{20ba^3} -$

$\dfrac{x^7}{112ba^5} - \dfrac{3x^9}{114ba^7}$ &c. but when $y = 60$, $x = 8{\cdot}6$ fere; consequently

when $- x$ in the corresponding abscissa $= 8{\cdot}6$, or $x = - 8{\cdot}6$, then the fluent or curve $= 43{\cdot}4$; but when $x = a$, and $x = - a$ in the abscissa, the fluent or curve $= 19{\cdot}6$, which added to the former is 63 fere, the length of the curve required.

IX. QUESTION 264, *by* Amicus.

To determine the greatest area that can be enclosed by a parabolic curve of the second kind, whose (equation is $ax^2 = y^3$, and) length 100 feet, and an ordinate rightly applied to its greatest axis?

Answered by Mr. Heath.

By the equation of the curve $axx = y^3$ (a being as yet unknown, but considered as fixed) we get $\sqrt{(\dot{x}\dot{x} + \dot{y}\dot{y})} = \dot{y} \sqrt{\dfrac{9y + 4a}{4a}}$ for

the fluxion of the curve, whose fluent (corrected) is $\dfrac{8a}{27} \left(\dfrac{9y + 4a}{4a}\right)^{\frac{3}{2}}$

$- \dfrac{8a}{27} = c = 50$. Whence $aa + \dfrac{27cc - 36yy}{16(c - y)} a = \dfrac{27y^3}{16(c - y)}.$

And $a = \dfrac{\sqrt{(27(64cy^3 - 16y^4 - 72ccyy + 27c^4)} + 36yy - 27cc)}}{32(c - y)}.$

The fluxion of the area of the semi-parabola $= y\dot{x} = \dfrac{3y\dot{y}\sqrt{y}}{2\sqrt{a}}$ whose

fluent $\dfrac{3\sqrt{y^5}}{5\sqrt{a}}$ must be a maximum, and consequently $\dfrac{y^5}{a}$; therefore

$\dfrac{8y^5 - y^6}{\sqrt{(3(64cy^3 - 16y^4 - 72ccyy + 27c^4))} + 12yy - 9cc}$ will be a

maximum. Put into fluxions, &c. $(5c - 6y) \times .$

$(\sqrt{(3(64cy^3 - 16y^4 - 72ccyy + 27c^4))} + 12yy - 9cc) - 3$

$$\times \frac{96cyy - 32y^3 - 72ccy}{\sqrt{(3(64cy^3 - 16y^4 - 72ccyy + 27c^4))}} + 24yy\,(c-y) = 0\,;\quad \text{here}$$

$y = 34\cdot89$ by a new method of solving equations, and consequently $a = 35\cdot543$ fere, and the area of the whole parabola formed thereby (which is now the greatest) $1447\cdot4$ fere.

In answer to the objections by *Amicus*, a, in the equation, is as much a variable quantity as x or y, till it is determined. And in the equation $\dfrac{8a}{27}\left(\dfrac{9y+4a}{4a}\right)^2 - \dfrac{8a}{27} = c$ (where $y =$

$\dfrac{a^{\frac{1}{3}}(27c+8a)^{\frac{2}{3}} - 4a}{9}$) it has a variable relation to y, when $y^5 \div a$

or $y \div a^{\frac{1}{5}}$ is to be determined a maximum; and substituting this

way for the value of y, $\dfrac{a^{\frac{1}{3}}(27c-8a)^{\frac{2}{3}} - 4a}{9a^{\frac{1}{5}}}$, or $a^{\frac{2}{15}}(27c+8a)^{\frac{2}{3}}$

$- 4a^{\frac{4}{5}}$, will as properly express the maximum, as if it had been denoted by relative y's; hence by making the fluxion of it $= 0$, we get by reduction $aa - \dfrac{9c}{16}a = \dfrac{27cc}{16^4}$, where the value of $a =$

$\dfrac{3c\sqrt{21} + 9c}{32} = 35\cdot5433$ &c. whence $y = 34\cdot8909$ &c. and $6y^{\frac{5}{2}} \div 5a^{\frac{1}{2}}$

$= 1447\cdot4001$, the area of the greatest parabolic space, as before. Also when $c = 10$, then $x = 6\cdot913$, $y = 6\cdot978$, and $a = 7\cdot1087$ &c. And when $c = 1000$, $x = 691\cdot3$, $y = 697\cdot82$, and $a = 710\cdot87$; by which it is proved that when the figure is a maximum, the abscissa and ordinate will be nearly equal.

In the present case, where $c = 50$, let $x = y$, then the equation $axx = y^3$ becomes $axx = xxy$ when the semi-parabola is nearly the greatest; and consequently $a = y = x$, at that time $= 27c \div (13^{\frac{2}{3}} - 8) = 34\cdot7292$ &c. from what is done above. Whence $1447\cdot34$ will be the area of the parabola very nearly a maximum. And therefore in all questions of this nature the area may be computed by taking the ordinates and abscissas equal; the error being inconsiderable in the maximum. Draw the parabola ABⁱcA correct, and it will approach the form of a quadrant aBkcfa, as near as the inequality of the curve Bⁱc permits; and its double will ever be inscribed in a segment of a circle something less than a semi-circle: but if the point c be made to pass through f, it will be inscribed exactly in a semi-circle, and the area of the semi-space ABⁱcA will vary from the true maximum, but by an exceeding small quantity, as is evident from above. When ABkcA is a maximum, the space ABⁱcA, or $\triangle$ ABC is a maximum, which is when AB $=$ AC :
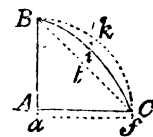
for, by the equation $x = \dfrac{y\sqrt{y}}{\sqrt{a}}$, and $yx = \dfrac{yy\sqrt{y}}{\sqrt{a}}$ or $\dfrac{y^5}{a}$ is a maximum

as proved before : but since the relation of y and a can be only had from the rectification of the curve Bia, and its equation, with the length of curvature given, the inequality of the *(fiddlestick)* space B$ic i$B, or unequal curvature of Bic, involves a necessity of some little inequality betwixt ʌB and ʌc, when ʌBicʌ is most capacious.

Remark. This question is the same as the 185th. the solution of which was not compleated. Both the above two methods of solution bring out true answers, but the latter is much the easier. By Mr. *Emerson's* method explained at the solution of the prize question for 1741 the same conclusions are also easily obtained.   H.

<center>THE PRIZE QUESTION.</center>

Suppose a cask composed of two equal conoids, formed by the revolution, about its axis of a curve whose equation is $y^y - 1000000x = 0$. Required the content of the cask if its length is 31·907 inches and bung diameter 14 inches?

<center>*Answered by Mr.* R. Heath.</center>

Projecting the curve ʌmnG, which is easily done by making $y =$ 1, 2, 3, 4, 5, &c. and thence finding the x's by the

equation $\dfrac{y^y}{1000000} = x$; whence it will appear that

when the semi-bung diameter $y = 7$ inches, that x will be but ·823543 which the half length of the cask should be taken out of, and proves an answer to be impossible : but correcting the data, and making the semi-bung diameter $y = 8 = $ BG, then $x = 16\cdot777216 = $ ʌB; whence taking 15·9535 (or 15·953673) the semi-length of the cask, and there remains ·823543 $=$ ʌa; whence $am = 7$ inches, or the head diameter $= 14$, bung diameter $= 16$, and length 31·907346 inches; and being near the form of a cylinder, the mean diameter $= 15$ inches, and content of the semi-cask by the rotation of B$oamn$G about aoB $= 2819\cdot25$ &c. or 10 ale gallons, and the whole cask 20 gallons.
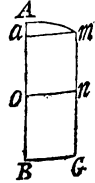
Questions proposed in 1746, and answered in 1747.

I.   QUESTION 265, by Mr. Heath.

If the hind wheel of a coach be seven feet in diameter, and a tack be driven into the middle of the spoke (or radius) standing next the ground, and a nail touch the ground at the end of the said spoke (or radius) when the coach sets out to travel: Quere how many miles will the tack and nail travel respectively in driving the coach from London to Exeter; allowing the distance between those two places to be 200 miles? What will be the nature of the curves they describe? And their position, or height of tack and nail from the ground, at the end of the journey.

Answered by the Proposer, Mr. Heath.
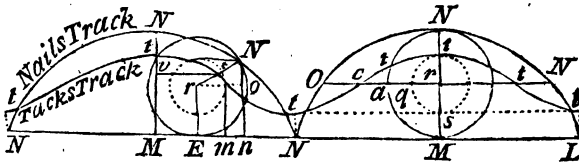
The track of the nail will be the curve LNNONN &c. which is a cycloid, and that of the tack the curve ttt &c. both which curves are thus rectified. Put $a = $ NM $ = 7$, $x = $ Nr, $y = $ or, $v = ar = \sqrt{(ax - xx)}$, $z = $ circular arch N$a = oa$ (per nature curve), $\therefore z + v = y$, $vv = ax - xx$, $\dot{v} = \dfrac{a - 2x}{2v} \dot{x}$, $\dot{z} = \dfrac{a\dot{x}}{2v}$, also $\dot{y} = \dot{z} + \dot{v} = \dfrac{a - x}{v} \dot{x} = \dfrac{a - x}{\sqrt{(ax - xx)}} \dot{x}$, hence $\sqrt{(\dot{x}\dot{x} + \dot{y}\dot{y})} = \dot{x}\sqrt{\dfrac{a}{x}}$, whose fluent is $2\sqrt{ax} = $ arch NO; therefore when $x = a$, arch NON $= 2a$, consequently NONN$tL = 4a = 28$ feet, the nail describes in one revolution of the wheel. The nature of the curve described by the tack is expressed by $cq = 2$ arch qt, which referred to the foregoing symbols will be $y = 2z + v$, parts of the lesser circle. Consequently, by substitution in this case, the fluxion of the inner curve ctt will be $\dfrac{\dot{x}}{2}\sqrt{\dfrac{9aa - 8ax}{ax - xx}}$; whose fluent, by series, is $\sqrt{ax} \times : 3 + \dfrac{x}{18a} + \dfrac{7xx}{216aa} + \dfrac{613x^3}{3 \times 7 \times 6^4a^3}$ &c. which when $x = ts = 3·5 = a$, is 11·696 fere (but this fluent may be otherwise found). Hence the

tack describes 23·392 feet in one revolution, whilst the nail describes
28. and the axis 21·9911485 &c. = 7 × 3·141592653 the wheel's
circumference; by which the wheel will revolve 48019·31995 times
in travelling 200 miles.——But 21·9911485 : 28 :: 200 : 254·648
miles, travelled by the nail. And 21·9911485 : 23·392 :: 200 :
212·74 miles travelled by the tack pretty nearly; the small difference
being only what EN (·31995 part of a revolution) differs in proportion
with N, N, and t, t, part of curves described by nail and tack at the
end of the journey, in the position of N, t, r, making an ∠NrE =
·31995 × 360 = 115° 11', or Nro = 25° 11' with the horizon.
Whence Nn = 4·988 feet, and tm = 4·244 the height of nail and
tack from the ground.

N. B. The number of whole revolutions multiplied into the whole
curves aforesaid, and the rectifications of the last parts being respec-
tively added, will exactly shew the distances described by the nail
and tack, very nearly as before. Or this question might be resolved
by a curve described upon the curve of a great circle of the earth,
which solution would come out not much different.

II. QUESTION 266, *by Mr.* Farrer.

Surveying a triangular field ABC, and standing at the corner c, I
took the angle included between the side BC and a line drawn from
the angle c, to a house situated within the field, and found it 78°
10'. I then proceeded to measure the shortest distance to the oppo-
site side AB, and having measured 20 chains, I observed the house
and the angle A in a right line; then measured on 10 chains to the
side AB: I likewise observed that the sides AC and BC were equal,
and the house equally distant from the angles A and c. Quere the
area of the field?

Answered by Mr. 1. Waine.

Put x = sine, and y = cos. ∠CAD = ∠CBD, and s and c for those
of the given ∠FCB. Then by the elements of
trigonometry, $sx - cy$ = sine, and $sy + cx$ =
cos. ∠FCD. But ∠ACD — FCD = ∠ACF whose

tang. is $\frac{syy - sxx + 2cxy}{cxx - cyy + 2sxy}$, which per quest.

= tang. ∠CAG. But AD : GD :: radius :

tang. ∠GAD = $\frac{x}{3y}$; whence $\frac{2xy}{3yy + xx}$ =

tang. ∠CAD — ∠GAD = $\frac{syy - sxx + 2cxy}{cxx - cyy + 2sxy}$; reduced, $\frac{x^4}{y^4} + \frac{6xx}{yy}$
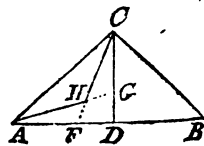
$-\dfrac{8cx}{sy} = 3$; solved, $\dfrac{x}{y} = \cdot 80907 = $ tang. of $38° 58' 30''$; whence AD $= 37\cdot 0796$, and consequently the area $= 111\cdot 239$ acres.

Mr. Ash *answers this Question thus:*

Call the tang. of the given angle m; CD, a; and GD, c; also put x for tang. $\angle$ ACD. Then tang. $\angle$ ACB will be $\dfrac{2x}{1 - xx}$, and that of $\angle$ ACH (when $\angle$ ACB is obtuse) $\dfrac{2x + m - mxx}{2mx + xx - 1}$. But (by trig.) $\dfrac{c}{ax} = $ tang. $\angle$ DAG, $\therefore \dfrac{ax - cx}{axx - c} = $ tang. $\angle$ HAC $= \angle$ ACH; consequently $\dfrac{ax - cx}{axx - c} = \dfrac{2x + m - mxx}{2mx + xx - 1}$; solved, $x = 1\cdot 2359 =$ tang. of $51° 1'$ fere. Whence the area of the field $= 111\cdot 23$ acres.

III.   QUESTION 267, *by Mr.* Powle.

Given $35x + 43y + 55z = 4000$, to find all the possible values of x, y, z in whole numbers, and to shew the method of investigation.

Answered by Mr. J. Waine.

For z in the given equation $(35x + 43y + 55z)$ substitute its least value, viz. 1, and we have $y = 91\frac{32}{43} - \frac{35}{43}x$, therefore $\dfrac{32 - 35x}{43}$ must be some whole number; multiplied by 2 and divided, is $1\frac{21}{43} - \dfrac{27x}{43} - x$, $\therefore \dfrac{21 - 27x}{43} \times 2$ will be a whole number $= \frac{42}{43} - \dfrac{11x}{43} - x$, $\therefore \dfrac{42 - 11x}{43} \times 4$ is also some whole number $= 3\frac{39}{43} - \dfrac{x}{43} - x$, whence $\dfrac{39 - x}{43}$, or rather $\dfrac{x - 39}{43} = m$, a whole number, $\therefore x = 43m + 39$, which value of x substituted for x in $y = 91\frac{32}{43} - \dfrac{35x}{43}$, gives $y = 60 - 35m$; whence $x = 43m + 39 = 82$ or 39, $y = 60 - 35m = 60$ or 35, $z = 1$. And thus by assuming $z = 1, 2, 3, 4$, &c. we obtain all the possible values of x, y, and z, viz. z admitting of 65 different values, and x and y of 89.

IV.   QUESTION 268, *by* Fortunatus.

Let the sorts of faces to be thrown on seven dice by four persons,

at a single throw each, be as follows, *viz.* by A, a^4bcd; by B, a^2bcdef; by C, $a^3b^2c^2d$; by D, a^3b^2cd: Quere their respective chances of winning? And what throw as to sorts of faces, has the greatest number of chances for coming up, at a single throw, of all the sorts which can be thrown on the said number of dice?

N. B. The number of the same and different letters represent so many of the same and different sorts of faces, *viz.* so many aces, duces, trays, cators, &c. of the same and different sorts.

Answered by Mr. Heath.

No solution has appeared in any author to questions of this nature, which will admit of several varieties still to be proposed. Mr. *Kay's* question was the first proposed, in a particular case, and the general method of solution is exhibited in the following examples, not hit upon before, that I have seen, by any.

Sorts of Faces.	No. of Chances.	Combinations.		Permutations of each comb.	
a^2	6	Sides revolve			dec. dice
ab	30	$= \frac{6.5}{1.2}$ Prog. of	$\times$	$\frac{2.1}{1.1}$	decrease of
6^2 all ch. on 2 dice	$= 36$	equal indices.			highest indices of faces.
a^3	6				
a^2b	90	$= 6.5$	$\times$	$\frac{3.2.1}{2.1.1}$	
abc	120	$= \frac{6.5.4}{1.2.3}$ Prog. eq. ind.	$\times$	$\frac{3.2.1}{1.1.1}$	
6^3 all ch. on 3 dice	$= 216$				
a^4	6				
a^3b	120	$= 6.5$	$\times$	$\frac{4.3.2.1}{3.2.1.1}$	
a^2b^2	90	$= \frac{6.5}{1.2}$	$\times$	$\frac{4.3.2.1}{2.1.2.1}$	
a^2bc	720	$= \frac{6.5.4}{1.2}$	$\times$	$\frac{4.3.2.1}{2.1.1.1}$	
$abcd$	360	$= \frac{6.5.4.3}{1.2.3.4}$	$\times$	$\frac{4.3.2.1}{1.1.1.1}$	
6^4 all ch. on 4 dice	$= 1296$				

Faces.	No. of Chances	Combinations		Permutations of each comb.
a^5	6			
a^4b	150	$= 6.5$	×	$\dfrac{5.4.3.2.1}{4.3.2.1.1}$
a^3b^2	300	$= 6.5$	×	$\dfrac{5.4.3.2.1}{3.2.1.2.2}$
a^3bc	1200	$= \dfrac{6.5.4}{1.2}$	×	$\dfrac{5.4.3.2.1}{3.2.1.1.1}$
a^2b^2c	1800	$= \dfrac{6.5.4}{1.2}$	×	$\dfrac{5.4.3.2.1}{2.1.2.1.1}$
a^2bcd	3600	$= \dfrac{6.5.4.3}{1.2.3}$	×	$\dfrac{5.4.3.2.1}{2.1.1.1.1}$
$abcde$	720	$= \dfrac{6.5.4.3.2}{1.2.3.4.5}$	×	$\dfrac{5.4.3.2.1}{1.1.1.1.1}$
6^5 all ch. on 5 dice	$= 7776$			
a^6	6			
a^5b	180	$= 6.5$	×	$\dfrac{6.5.4.3.2.1}{5.4.3.2.1.1}$
a^4b^2	450	$= 6.5.4$	×	$\dfrac{6.5.4.3.2.1}{4.3.2.1.2.1}$
a^4bc	1800	$= \dfrac{6.5.4}{1.2}$	×	$\dfrac{6.5.4.3.2.1}{4.3.2.1.1.1}$
a^3b^3	300	$= \dfrac{6.5}{1.2}$	×	$\dfrac{6.5.4.3.2.1}{3.2.1.3.2.1}$
a^3b^2c	7200	$= 6.5.4$	×	$\dfrac{6.5.4.3.2.1}{3.2.1.2.1.1}$
a^3bcd	7200	$= \dfrac{6.5.4.3}{1.2.3}$	×	$\dfrac{6.5.4.3.2.1}{3.2.1.1.1.1}$
$a^2b^2c^2$	1800	$= \dfrac{6.5.4}{1.2.3}$	×	$\dfrac{6.5.4.3.2.1}{2.1.2.1.2.1}$
a^2b^2cd	16200	$= \dfrac{6.5.4.3}{1.2.1.2}$	×	$\dfrac{6.5.4.3.2.1}{2.1.2.1.1.1}$
a^2bcde	10800	$= \dfrac{6.5.4.3.2}{1.2.3.4}$	×	$\dfrac{6.5.4.3.2.1}{2.1.1.1.1.1}$
$abcdef$	720	$= \dfrac{6.5.4.3.2.1}{1.2.3.4.5.6}$	×	$\dfrac{6.5.4.3.2.1}{1.1.1.1.1.1}$
6^6 all ch. on 6 dice	$= 46656$			

Faces	No. of Chances	Combinations		Permutations of each comb.
a^7	6			
a^6b	210	$= 6.5$	$\times$	$\frac{7.6.5.4.3.2.1}{\ldots 4.3.2.1.1}$
a^5b^2	630	$= 6.5$	$\times$	$\frac{7.6.5.4.3.2.1}{5.4.3.2.1.2.1}$
a^5bc	2520	$= \frac{6.5.4}{1.2}$	$\times$	$\frac{7.6.5.4.3.2.1}{5.4.3.2.1.1.1}$
a^4b^3	1050	$= 6.5$	$\times$	$\frac{7.6.5.4.3.2.1}{4.3.2.1.3.2.1}$
a^4b^2c	12600	$= 6.5.4$	$\times$	$\frac{7.6.5.4.3.2.1}{4.3.2.1.2.1.1}$
a^4bcd	12600	$= \frac{6.5.4.3}{1.2.3}$	$\times$	$\frac{7.6.5.4.3.2.1}{4.3.2.1.1.1.1}$
a^3b^3c	8400	$= \frac{6.5.4}{1.3}$	$\times$	$\frac{7.6.5.4.3.2.1}{3.2.1.3.2.1.1}$
$a^3b^2c^2$	12600	$= \frac{6.5.4}{1.2}$	$\times$	$\frac{7.6.5.4.3.2.1}{3.2.1.2.1.2.1}$
a^3b^2cd	75600	$= \frac{6.5.4.3}{1.2}$	$\times$	$\frac{7.6.5.4.3.2.1}{3.2.1.2.1.1.1}$
a^3bcde	25200	$= \frac{6.5.4.3.2}{1.2.3.4}$	$\times$	$\frac{7.6.5.4.3.2.1}{3.2.1.1.1.1.1}$
$a^2b^2c^2d$	37800	$= \frac{6.5.4.3}{1.2.3}$	$\times$	$\frac{7.6.5.4.3.2.1}{2.1.2.1.2.1.1}$
a^2b^2cde	75600	$= \frac{6.5.4.3.2}{1.2.1.2.3}$	$\times$	$\frac{7.6.5.4.3.2.1}{2.1.2.1.1.1.1}$
a^2bcdef	15120	$= \frac{6.5.4.3.2.1}{1.2.3.4.5}$	$\times$	$\frac{7.6.5.4.3.2.1}{1.1.1.1.1.1}$
6^7 all ch. on 7 dice	$=279936$	Sum, &c. *ad infinitum.*		

Hence the respective chances of A, B, C, and D winning are obvious, as 12600, 15120, 37800, and 75600, the same as 1, 1·2, 3, and 6 exactly. And a^3bbcd and $aabbcde$ have equal, and the greatest number of chances for coming up, *viz.* 75600. Hence also is inferred that the best throw for winning will be when the sorts are within a place or two of being all different.

V. QUESTION 269, *by Mr. J. Ash.*

A gentleman has a piece of ground, whose three sides are an abscissa, semi-ordinate, and curve, the equation of which is $ax = y^3$; he has taken from thence the biggest oblong garden which he could possibly enclose, whose area is 141·84243 poles; and he finds the abscissa longer than the semi-ordinate by 5 poles. It is required from thence to find the area of the whole enclosure, its perimeter, and the sides of the garden (taken out of it) separately?

Answered by Mr. John Turner.

Put $x =$ HG, $y =$ GF, $b =$ AD, $b - x =$ GD, $a =$ parameter, the equation of the curve being $ax = y^3$, $\therefore y = \sqrt[3]{ax}$, and $(b - x)\sqrt[3]{ax}$ is to be a maximum; which in fluxions, &c. gives $-\sqrt[3]{ax} + \dfrac{b - x}{\sqrt[3]{aaxx}} \times$

$\dfrac{a}{3} = 0$; hence $3x = b - x$, and $x = \frac{1}{4}b$ exactly.
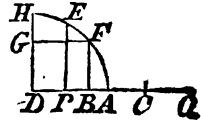
To proceed, put $z =$ HD, $\frac{1}{4}z =$ HG, $z - 5 =$ DA, $\frac{3}{4}z =$ GD, $m = 141.84243$; then GF $= \sqrt[3]{(\frac{1}{4}az)}$, DA $= \sqrt[3]{az} = z - 5$, $d = \sqrt[3]{4} = 1.5874$, and per quest. $\sqrt[3]{(\frac{1}{4}az)} \times \frac{3}{4}z = m$. Hence $\sqrt[3]{az} = 4dm \div 3z = z - 5$; $3zz - 15z = 4dm$; here $z = 20$ exactly. And $a = 168.75$; also the length of the curve HFA $= 27.014$. So that the fences of the garden are GF $= 9.4494$, and GD $= 15$: Moreover HD $= 20$, DA $= 15$. And the area HFADH $= \frac{1}{4}$HD $\times$ DA $= 225$ square poles, or 1 acre, 1 rood, 25 perches.

VI.　QUESTION 270, by Rhinoceros.

The perpendicular of a triangular field is 200 poles; the line equally bisecting the angle opposite to the base, drawn to the base, is 250 poles; and the distance from the said obtuse angle to the middle of the base is 295 poles: Quere the sides and area of that triangular field, with the geometrical construction of the same?

Answered by Mr. Farrer.

At the point D, on the line AD raise the perp. CD $= 200$ poles; make CE $= 250$, and CF $= 295$; produce CE to cut the perp. FG in G, draw CH making the $\angle$GCH $= \angle$CGH; then upon H, as a centre, with the rad. HG describe the arch CBGA. Join CB and CA, and the $\triangle$ABC is that required.
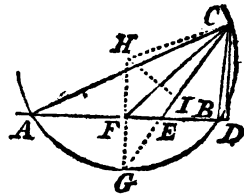
Calculation. Having CD, CE, and CF given, find ED $= 150$, FD $= 216.85$, and EF $= 66.15$. But ED : DC :: EF : FG $= 89.14$; $\therefore$ GE $= 111.42$; and GF : GE :: GI : GH $= 225.89$; hence AB $= 359.58$, BC $= 203.41$, and AC $= 441.21$. The area 31958 square poles, or 224 acres 2 rood 38 perches.

VII.　QUESTION 271, by Mr. Christ. Mason.

Required the time of the year when there is the shortest twilight in lat. 50° 46′?

Answered by Mr. J. Turner.

If κ represent the north pole, z the zenith, H⊙ an arch of the horizon, A⊙A a parallel of the sun's depression 18° below it; and ⊙⊙ a parallel of declination descended; then the angle ⊙N⊙ will be a minimum : but 'tis also evident that when the crepusculum is the shortest, the distance ⊙⊙ descended in the parallel of declination will be a minimum : being the arch of a lesser circle cutting the same azimuth in the points of setting and end of twilight. And this will be when the motion of the sun is most perpendicular, and therefore descends the fastest to 18°, or the parallel of A⊙A ; which he does by touching at the points ⊙ and ⊙ in the same azimuth z⊙⊙ with his setting, instead of making an oblique angle with the setting azimuth, as in the position of ⊙m⊙ with za⊙. This being admitted, put $y = $ cos. N⊙ = N⊙ ; m and n sine and cos. z⊙ = 108°, or sines of 72° and 18° ; p and q sine and cos. Nz = comp. lat. and rad. unity = z⊙.
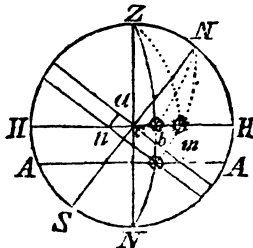
Then per spherics $\dfrac{y}{p} = $ cos. Nz⊙, and also $\dfrac{nq - y}{mp} = $ cos. Nz⊙, which (for the above reasons) are = to each other, i. e. $y \div p = \dfrac{nq - y}{mp}$, therefore $y = \dfrac{nq}{m + 1}$, which is a general theorem for all questions of this nature. And the sun's declination in the present case is found by it = 7° 2′ 42″ southerly, answering to Feb. 19, or Sept. 30 ; $\therefore m + 1 : n :: q : y$.

N. B. If the lat. is north, the declination must be south, and vice versa.

Mr. Farrer *answers thus.*

Let ⊙ be the point in the circle bounding twilight 18° below the horizon, where the shortest twilight happens (without saying why the bounding point falls in the same azimuth circle with the sun's setting.) Then in the triangle bc⊙, $\angle$⊙ = 90°, $\angle c = $ co-lat. ⊙b = 9°, c⊙ = cn, whence rad. : tang. ⊙b :: sine lat. : sine arc = 7° 3′ fere, the sun's declination.

A construction to this question is given at question 564.

VIII. QUESTION 272, *by* F. R. S.

In what law of gravity will a projectile describe a curve expressed by the equation $ax^2 = y^3$, in a non-resisting medium ?

Answered by Mr. Landen.

Let A*r*c represent the curve; AB the axis thereof; B*m* and

$\text{H}r$ ordinates indefinitely near each other. Call $A\text{H}$, x; $\text{H}r$, y; and the gravity G. Then since the velocity in the direction $\text{H}r$ is always the same, that in the direction rn will be $\dfrac{\dot{x}}{\dot{y}}$, whose fluxion $\dfrac{\ddot{x}}{\dot{y}}$ ($\dot{y}$ being constant) will be as $G \times \dot{y}$; that is, as the force by which the body is accelerated at r drawn into the time of describing rm. Hence putting $G \times \dot{y} = \dfrac{\ddot{x}}{\dot{y}}$ we have $G = \dfrac{\ddot{x}}{\dot{y}\dot{y}}$. In which expression if for $\ddot{x}$ we put its value $\dfrac{15y\sqrt{\dot{y}}}{4\sqrt{a}}$, found by the equation of the curve ($axx = y^3$) we get $G = \dfrac{15\sqrt{y}}{4\sqrt{a}}$; i. e. the gravity in this case must be in a subduplicate ratio of the ordinates, or in a subquintuplicate ratio of x, the distance of the ordinate from the vertex.
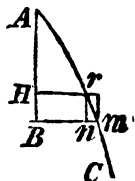
IX. QUESTION 273, by Mr. Farrer.

Quere the area of a right-angled triangle whose hypothenuse is x^{3x}, and the two legs x^{2x} and x^x?

Answered by Mr. I. Ash.

Let $y = x^x$, then (per 47 Euc. I. and per quest.) $y^2 + y^4 = y^6$, $y^2 = \dfrac{\sqrt{5}+1}{2} = 1{\cdot}618034$, $y = 1{\cdot}272019$, and the area required $= 1{\cdot}02908$.

Remark. Of this triangle, the perpendicular from the right angle on the hypothenuse is $= 1$. For it is $=$ the double area divided by the hypothenuse $=$ the product of the two legs divided by the hypothenuse $= \dfrac{x^x \times x^{2x}}{x^{3x}} = \dfrac{x^{3x}}{x^{3x}} = 1$. H.

X. QUESTION 274, by Mr. Clarke.

The thickness of a ring belonging to a ship's anchor is nine inches in circumference, and the outward circumference shewing the width of that ring is 50 inches? Quere the solid content, and weight thereof?

Solved by Mr. Landen.

Put $d =$ diff. of the given circumferences, $r =$ rad. of the lesser

circumference, and $x =$ any abscissa of the circle whose radius is r. Then $d \times 2\sqrt{(2ax - xx)} \dot{x}$ is the fluxion of half the required solidity. But the fluent of $2\sqrt{(2ax - xx)} \dot{x}$, when $x = r$, is the area of the semi-circle, whose radius is r; therefore the area of the circle shewing the thickness of the ring, multiplied by d, the diff. of the given circumferences, will give the solid content of the ring $= 264$ inches, fere. Whence the weight thereof, according to Dr. Wiberd, is nearly $= 77$ pounds avoirdupois.

XI. QUESTION 275, *by* Filius Diophanti.

To find three numbers, that when each is subtracted from the cube of their sum, a cube number shall remain ?

Answered by Mr. J. Hampson.

The numbers are $\frac{13551}{85184}$, $\frac{19467}{85184}$, and $\frac{18954}{85184}$; these fractions are in lower terms than those given by Dr. Wallis from Dr. Pell, where the method of solution may be seen.

Mr. Turner *gives this Answer from Dr. Wallis's Algebra.*

1st. $a = \frac{494424}{2352637}$; 2d, $b = \frac{472696}{2352637}$; 3d, $c = \frac{448000}{2352637}$, whose sum $= \frac{1415120}{2352637} = \frac{80}{133}$. The cube of their sum $= \frac{512000}{2352637}$, from which numbers taking severally the values of a, b, and c, there will remain these three cubes, viz. $\frac{17576}{2352637}$, $\frac{39304}{2352637}$, and $\frac{64000}{2352637}$, whose roots are $\frac{26}{133}$, $\frac{34}{133}$, and $\frac{40}{133}$.

Remark. This is the same with question 51.

XII. QUESTION 276, *by Mr.* J. Landen.

It is required to find the periodic time of a pendulum describing a conical surface; the perpendicular height of the described cone being two hundred inches ?

Answered by Mr. I. Ash.

Suppose the sine of the vertical angle of the cone ·0507; then by trigonometry the side or length is found $= 3944·77$ inches or $328·73$ feet $=$ length of the pendulum's string. Hence (per Keil's Introd. Theor. 11, p. 302), the time of one revolution is equal to the time

of the perpendicular fall of a heavy body from a height equal to the pendulum's length, ∴ $16\frac{1}{12}$: $1''$ :: $328.73 : 20.439442''$ the square of the time, whose root $= 4.521$ seconds required.

Remark. The principle used in the above solution is not general for any angle at the vertex, but only for that one particular angle there used, as may be seen in the Prop. of Keil there referred to. But the times of gyration do not depend on the length of the string but only on the altitude of the cone, they being universally as the square roots of the altitudes; and when the altitudes are equal, the times will be equal also, whatever the length of the pendulums may be.

By Prop. IX. Emerson's Centrip. Forces, the proportion is universally thus, as $\sqrt{16\frac{1}{12}}$ feet : $\sqrt{400}$ inches (twice the given $\perp$) ::

$3.1416 : 3.1416\sqrt{\dfrac{400}{193}} = \dfrac{62.832}{\sqrt{193}} = 4.5228$ seconds, the time required. H.

XIII. QUESTION 277, *by* Crocus Metallorum.

What annuity, to continue as many years as its pounds, can I purchase for the square root of its pounds ready money, allowing me 5l. per cent. per ann. compound interest for my bargain?

Answered by Mr. Heath.

Put $a =$ the pounds of an annuity, $r = 1.05$ the amount of a pound and its interest for one year at 5 per cent. $t =$ the year's continuance; then $\dfrac{a}{r-1} - \dfrac{a}{r^a(r-1)} = z$, the present worth.

And if the conditions of the question be substituted therein, the equation becomes $\dfrac{a}{a-1} - \dfrac{a}{r^a(r-1)} = \sqrt{a}$. Whence 1.05^a

$= \dfrac{\sqrt{a}}{\sqrt{a} - .05}$; here $a = 1.034$ fere $= 1l.$ 0s. 7d. the annuity required.

XIV. QUESTION 278, *by* Hurlothrundro.

Required the ratio of the diameter of the bore to the length of a piece of cannon (or other fire arms) to make it capable of throwing a ball the farthest possible; supposing the diameter of the ball nearly equal to the diameter of the bore, with a proportionable weight of powder, and the metal of the piece formed sufficient to sustain the effect?

Observations on this Question by the Editor.

This question was proposed with an intent to improve gunnery, of which there are several things wanting. In particular, a treatise on the subject by an experienced hand : For to be treated on by any other person, will only be compiling of matters already known. And in order for this improvement, and the solution of this question, experiments should be made in a general way, which we have not yet received.

Remark. The Editor observes above that this question was proposed with an intent to improve practical gunnery, though it does not appear from the nature of the science that it would at all have answered that purpose. It is well known that *short pieces* are requisite at *sea*, both for the convenience of working them in an engagement, and on account of the small space they must stand in when the ports are closed. The *land service*, on the contrary, requires *long pieces*, particularly in the attack of a place, in order to preserve the embrasures from the blast of the powder, which short pieces would soon destroy, besides the danger of setting fire to a fascine battery. Not only the *lengths* of pieces are *limited* by the nature of the service, but also the *diameters of the bores*; for pieces which carry balls from 24 to 42 pounds comprehend the limits of *battering* cannon, and those from 3 to 12 pounds limit *field* pieces. Experience proves that balls of a less weight than 24 pounds are insufficient to make a moderate breach ; and that pieces carrying 42 pound balls, or upwards, become unmanageable from their great weight; so that in general the 32 pounder is the most common battering piece. When the field piece exceeds 12 pounds, it, in like manner, becomes too unwieldy for *that* service.

Mr. *Robins* is the only author that I know of, who has solved a proposition exhibiting the relation between the *velocity* of the ball and the *dimensions* of the piece. Another author, who proceeds in a very different manner, makes the *velocity* always increase with the *length* of the piece.

If any person, however, think it worth his while to go through the calculation of this problem, he may easily do it, making the expression for the velocity found by prop. 7, of *Robins's* Principles of Gunnery, a maximum, then its fluxion being taken, &c. there will be determined the relation between the diameter and length. H.

Additional Remarks. What could not be accomplished, with respect to this question, by Mr. Heath in 1747, nor by Dr. Hutton in 1774, we are now happily enabled to do, by the aid of those very extensive and accurate gunnery experiments, lately published by the latter author, in the 2d and 3d vols. of his Tracts. We there find, particularly in the Tablet, p. 310, vol. 2, what are the quantities or the lengths of the charges, for all kind of guns, in terms of their calibers or diameters, to discharge their balls with the greatest velocity. This of course answers one part of the enquiry ;

for all guns, whether great or small, from the largest cannon to the musket, discharge their balls with equal velocity, when they are formed of similar dimensions, and similarly loaded, that is, the like parts measured by the same number of calibers. Thus then the greatest velocity is determined for any piece of ordnance, which is perhaps all that the proposer meant by his problem ; for he does not appear to be aware that, with the same initial velocities, the ranges will be greater, as the ball is larger, without end, (when of the same kind of matter), nor even that these initial velocities are all equal, as above mentioned. L.

THE PRIZE QUESTION, *by Mr.* W. Chapple.

A gentleman has a circular garden, whose diameter is 310 yards, in which is contained a circular pond, whose diameter is 100 yards, so situated in respect of each other, that their peripheries will inscribe and circumscribe an infinite number of triangles (*i. e.* whose sides shall be tangents to the pond, and angles in the fence of the garden.) He being disposed to make enclosures for different uses, and farther ornaments on his scheme begun, in order thereto applies himself to the artists of Great Britain for the dimensions of the greatest and least triangles that can be inscribed and circumscribed as aforesaid ? and the nearest distance of the peripheries of the garden and pond ? and for a demonstration of the truth of the pond's situation ?

Answered by Mr. R. Heath.

The biggest two circles, within one another, admitting of an infinite number of triangles to be drawn with their angles and sides terminating in their outer, and touching their inner peripheries, are those which are concentric, and their diameters exactly as 2 to 1 ; in which case the triangles will be all equal and equilateral. This is so evident as to appear upon the slightest examination. And if the diameters be in proportion less than 2 to 1, then no triangle whatsoever can be drawn as aforesaid. But if the proportion be greater than 2 to 1 (as 310 to 100, or 3·1 to 1 as in the present case) their peripheries being eccentric and at a proper distance, will admit of an infinite variety of triangles to be drawn in and about them, from the isosceles △ ACC to that of BDD, which are the greatest and least triangles : Because the area of every triangle so drawn, being equal to the sum of all the sides into half the given radius of the inscribed circle, the more the sides of any triangle are situated about the centre, or diameter, or the farther removed from them, the greater or less will the periphery, and consequently area, of that triangle be : the sides of the triangles ACC and BDD being the most near and remote to such a situation.

For the property of drawing triangles as aforesaid, in and about eccentric circles, there is this demonstration.

First to inscribe the isosceles, or one △, Acc in the greater circle, which at the same time shall circumscribe the lesser circle. Put m = rad. greater circle = Aa, n = rad. lesser = ov, and x = AO; then $\sqrt{(xx - nn)}$ = Av; per sim.

$$\triangle s, \text{ Av : Ao :: A}t \text{ : AC} = \frac{x + n}{\sqrt{(xx - nn)}}$$

× x; again, AO : Av :: AB : AC = $\frac{2m\sqrt{(xx - nn)}}{x}$; hence $\frac{x + n}{\sqrt{(xx - nn)}}$ × x

$= \frac{2m\sqrt{(xx - nn)}}{x}$, and $x - m = \sqrt{(m^2}$

$- 2mn) = ao$, shewing the distance of the centres to be least, or 0, when $2n = m$; both circles being then concentric, and radius of the greater = twice the radius of the less. Whence 12·269074 = Bt the nearest distance of the peripheries. And area △ Acc = 18242·89 yards, the greatest; and that of △ BDD = 16799·82, &c. the least.

The distance of these centres of the circles being known, as $d = co$ capable of having one triangle drawn as aforesaid: Suppose any chord AC = x touch the inner circle at v, then it is plain another chord can touch it somewhere at t, and if another can touch it at r, the property is proved. But, in general, the $\angle$ ABC = $\angle$ ADC = $\angle noc$ its sine = $x \div 2m$; and $\angle rBC = \angle cBt = \angle nDc$ (per 33 and 20 Euc. III. = $\dfrac{x}{2\sqrt{(2mn + m\sqrt{(4mm - xx)})}}$; the comp. or $\angle rcs$

= $\angle tcB = \angle ncD$, its sine = $\dfrac{x}{\sqrt{(2mm \times m\sqrt{(4mm - xx)})}}$; whence

by trig. $rB = Bt = \dfrac{2nm + 2n\sqrt{(4mm - xx)}}{x}$. Again $io = $

$\frac{1}{2}\sqrt{(4mm - xx)} - n$, and (per 47 Euc. I.) $ci = vn = \sqrt{(dd - (\frac{1}{2}\sqrt{(4mm - xx)} - n)^2})$; hence $vc = \frac{1}{2}x + vn$, A$v = $ A$r = \frac{1}{2}x - vn$, and AB and BC, also $\angle vcc = \angle cct$, and $\angle vct$ are expressed. Say $x \div 2m$ (s. $\angle$ ABC) : x (AC) or 1 : $2m$ :: s. $\angle vct$: AB, which gives an equation shewing the general value of x, whilst two other chords are tangents likewise; and triangles ilk, gcf, will revolve as in the 1st fig. Hence, if any chord or side of a triangle is given, the others follow by trigonometry; having first found the distance of the centres aforesaid. This property of drawing triangles about circles I discovered some years ago, as may be seen in the Monthly Oracle; though the proposer has greatly deserved in a long account of it from Scilly, printed in a book called the Quarterly Miscellanea Curiosa.

Mr. *Landen* puts b = rad. lesser circle, a = diam. greater, and x = altitude isosceles triangle, and gets $x = \dfrac{a + 2b}{2} \pm \sqrt{\dfrac{aa - 4ab}{4}}$
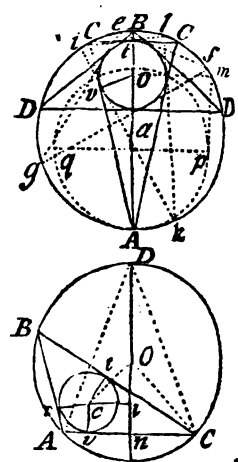
$= 297 \cdot 33$ or $112 \cdot 67$ &c. Whence he infers this construction (*see fig.* 1.) From the centre of the garden, along the diameter, set off the diameter of the pond, find a centre (at *n*) betwixt that point and the other end, describing a circle Apq, draw *pm* parallel, and *mo* perpendicular to AB, and the point *o* will be the centre of the pond.

Questions proposed in 1747, *and answered in* 1748.

I. QUESTION 279, *by Mr.* Landen.

A charming brisk maid has assur'd me and said,
Since I'm such a fine mathematician,
(Laying puzzling aside, for the joys of a bride)
She will wed me——but on this condition :
That I first shall unfold what * pieces of gold
Her father has for her in store :
And these I must find from the data subjoin'd,
And then I'm to puzzle no more.

* The pieces are half guineas, guineas, moidores, and three pound twelves. The whole number is 4000. And if v, x, y, and z be put for the number of each sort respectively, $v^4x^3y^2z$ is a maximum. Quere what is the lady's fortune?

Answered by Mr. J. Turner.

Let $v =$ the half guineas, $x =$ guineas, $y =$ moidores, and $z =$ 3l. 12s. By the question $v + x + y + z = 4000 = b$: Now $v^4x^3y^2z$ being a maximum, or (expunging z) $b - v - x - y = \dfrac{1}{v^4x^3y^2}$. In

fluxions $- \dot{v} - \dot{x} - \dot{y} = \dfrac{-2\dot{y}}{y^3x^3v^4} - \dfrac{3\dot{x}}{y^2x^4v^4} - \dfrac{4\dot{v}}{y^2x^3v^5}$; whence $\dot{y} =$

$\dfrac{2\dot{y}}{y^3x^3v^4}$; $\dot{x} = \dfrac{3\dot{x}}{y^2x^4v^4}$; $\dot{v} = \dfrac{4\dot{v}}{y^2x^3v^5}$. And $\dfrac{1}{v^4x^3y^2} = \dfrac{y}{2} = \dfrac{x}{3} = \dfrac{v}{4} =$

$b - v - x - y$. Consequently $x = \frac{3}{2}y$; $v = \frac{4}{2}x = 2y$: Also $\frac{1}{2}y$ $= b - 2y - \frac{3}{2}y - y$; or $y = \frac{1}{5}b$; $z = \frac{1}{10}b$; $x = \frac{3}{10}b$; $v = \frac{4}{10}b$; and the lady's fortune is 4620l. Hence this «

General Rule.

Make the sum of the exponents a denominator, and each particular exponent (of the quantities) the numerator of a fraction, these respectively multiplied into the whole number of pieces, will shew the particular number of pieces of each sort.

II. QUESTION 280, *by Mr.* John Williams.

Going along a river's side, on an even direct road ABC, I observed a tower on the other side of the river, whose angle of altitude at A was 5° 24′; going farther on to B, 100 yards, the angle of altitude was 6° 27½′; and intending, again, to take an observation when directly opposite to the tower, but was prevented by an island in the river (over-grown with furz), I then came to C, 400 yards from B, where I found the angle of altitude was 8° 36′. Quere the tower's height?

Answered by Mr. James Terey.

Put x = tower's height, AC = 500 = b, AB = 100 = d, BC = 400 = a. Co-sec. $\angle$ A = t, of B = v, of C = y. Then (rad. = 1) tx = AT, vx = BT, and yx = CT. Now (geometrically) making $t : v :: $ AI : IB :: AK : KB, and describing the semi-circle ITK ; also $v : y ::$ BL : LC :: BM : MC, describing the semi-circle LTM ; the intersection T, being the place of the tower, (for which see Universal Arith. prob. 26) let down TF $\perp$ AC. Per fig. $b : tx + yx$ :: $tx - yx : \dfrac{ttxx - yyxx}{b} = $ AF $-$ FC; also BT^2 $-$ BF^2 = AT^2 $-$ AF^2;
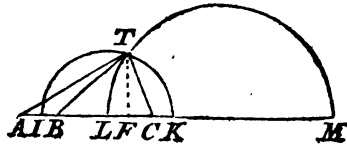
whence $attxx + dyyxx - bvvxx = abd$, and $x = \sqrt{\dfrac{abd}{att + dyy - bvv}}$

= 44·4609 &c. yards, the tower's height.

The same answered by Mr. J. Ash.

Call the co-secants of the three angles of altitude a, b, and c; AB, s; BC, $4s$; the tower's height, x. By trigon. (T being the top) AT = ax; BT = bx; CT = cx; cos. $\angle$CBT = $\dfrac{ccxx - bbxx - 16ss}{8sbx}$;

cos. $\angle$ABT = $\dfrac{ss + bbxx - aaxx}{2sbx}$; which two last expressions made

equal, $x = \sqrt{\dfrac{20ss}{cc - 5bb + 4aa}} = 44\cdot4557$ yards, the tower's height required.

A Geometrical Solution, by Caput Mortuum, taken from the London Magazine Improved, 1784.

Analysis. Let A, B, C, be the places of observation, D the foot of the tower; and suppose a circle to be described on the horizon with the radius DE, = the height of the tower. Make the angles ADE, BDF, CDG right angles, and draw AE, BF, GC. Then because the angles DAE, DBF, DCG, are given by the question, the triangles DAE, DBF, DCG, are given in species; and, moreover, since the side DE, (DF, DG) is common to all the triangles, the ratio of the sides AD, BD, CD is given. Draw BN, NS, parallel to CD, BC respectively; then NS = BC, and NB = SC; consequently, by similar triangles, AB : NS (BC) :: NB (SC) : SD; and hence we have this
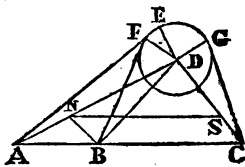
Construction. Having on a base DE assumed at pleasure, made three right-angled triangles (EDA, FDB, GDC) so that the vertical angles (EAD, FBD, GCD) are = 5° 24′, 6° 27½′, and 8° 36′, respectively; divide DA and DC, in N and S, in the given ratio of AB : BC (1 : 4). With ND, NB (SC) and DB constitute a triangle DNB; join AB and make the angle DAE = 5° 24′; also draw DF perpendicular to AD. Then it is evident from the analysis, that AB : DE :: 100 : the height of the tower.

Calculation. Suppose DE = 1; then DA = 10·5789, DB = 8·833. DC = 6·6122 the natural cotangents of the angles EAD, FBD, and GCD. Consequently in the △BND, there are given the three sides, DN = 8·4631, BN = 1·3224, and BD = 8·833, from whence the ∠BND will be found = 101° 59′ 20″, the supplement of which is 78° 0′ 40″ = ∠BNA. Then in the △BNA there are given two sides and the included angle, from whence AB will be found = 2·25. Therefore, 2·25 (AB) : 1 (DE) :: 100 : 44·44 &c. yards, the tower's height.

III. QUESTION 281, by Mr. J. May, jun. of Amsterdam.

It is required to find (by a general theorem) the number of fractions of different values, each less than unity, so that the greatest denominator be less than 100?

Answered by Mr. Heath.

The number of fractions of different and like values, each being less than unity, and the greatest denominator being any number, (from 2 upwards) will appear by the following series, continued to $\frac{98}{99}$, &c.

Sum = 27 = fractions of diff. values.		Sum = 9 = fractions of like value.
6	$\frac{8}{9}$ $\frac{7}{9}$ $\frac{6*5}{9}$ $\frac{4}{9}$ $\frac{3*2}{9}$ $\frac{1}{9}$	2
4	$\frac{7}{8}$ $\frac{6*5}{8}$ $\frac{4*3}{8}$ $\frac{2*1}{8}$	3
6	$\frac{6}{7}$ $\frac{5}{7}$ $\frac{4}{7}$ $\frac{3}{7}$ $\frac{2}{7}$ $\frac{1}{7}$	0
2	$\frac{5}{6}$ $\frac{4*3}{6}$ $\frac{*2*1}{6}$	3
4	$\frac{4}{5}$ $\frac{3}{5}$ $\frac{2}{5}$ $\frac{1}{5}$	0
2	$\frac{3}{4}$ $\frac{2*1}{4}$	1
2	$\frac{2}{3}$ $\frac{1}{3}$	0
1	$\frac{1}{2}$	0
27	36	9

Here it is evident, that the greatest numerator, or greatest denominator less unity, will be the number of terms, which will always be equal to the last term; therefore $\frac{n+1}{2} \times n =$ the sum of the series, or the number of different and like fractions in all cases. The fractions of like value with some of the rest, are all those not in their lowest terms: As those of different value are all the incommensurable ones: For the more speedy determining of which, all the incommensurable denominators, from the least in the series to the greatest, must be taken, viz. 3, 5, 7, 11, 13, &c. to 97, which are 25; and the sum of their different fractions will be the series 2, 4, 6, 10, 12, &c. to 96; each row being one less than the denom. To these if the different fractions (in lowest terms) of the commensurable denominators 2, 4, 6, 8, 10, &c. to 99, be added, the sum will be all the different fractions. But as there is more trouble than art to discover them, I shall leave it to persons of leisure to pursue the computation, the method being here planned out.

Mr. Ash, upon the same principles, computes the number of different fractions to be 3055; but doubts the truth.

N. B. By the above theorem, the whole number of fractions = 4851 (as Mr. Bamfield made it) from which subducting those of like values, all the different fractions will remain.

A general method for Solving this Question from the Diary for 1751, by Mr. Flitcon.

II.

Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	
2	P		1 16		1 2	8 30	2·35	8 16	20 24		8 4 30	72 73	P	28 57	24 66	2·43	24 72				42 56

(rotated numeric table — component primes, denominators and "Diff. fractions" columns)

The sum of all the different fractions.

3008	
774	774
999	666
564	564
416	416
314	314
198	198
71	71

Third columns collected.

Each diff. let. in 1st scheme denotes a diff. prime.

Steps	Denominators	Each single prime	The prod. of each two	Product of each three	Product of each four	Numb. of diff. fract. in each series.
1	x	$x-1$				$\ldots x^n$
2	x^2	x^2-x				$\ldots x^{2n}$
3	x^3	x^3-x^2				
4	vx	$vx-v+1$	vx			$\ldots m$
5	v^2x	v^2x-v^2	vx			$\ldots v^2x$
6	v^2x^2	$v^2x^2-v^2$	vx^2			$\ldots v^2x^2$
7	vxy	$vxy-vxy$	vxy	$vz+yz+vz$	-1	$\ldots m$
8		$-xy-yz+x$	$-yz+zv+v$	$ay+zu-v$	$-x+1\ldots m$	

$1\times$
$2\times$
$4\times$
$5\times$

N. B. P signifies each prime number in the second column in scheme 2, with a new number to each in the 3d column, deduced from the series in the first scheme, according to the general method : the different denominators being placed in the first column all along.

For the multipliers, or powers of each. Against 3 in the 1st column, stands 2 in the 3d column, each being drawn into 3 gives 6 to be set in the 3d column, against 9 in the first collection. Against 6 in the 1st col. is set 2·3 in the 2d col. whence (by 4th step) is found 2 for the 3d collection. Also for its multiples, into either, or both its parts, as 2 × 3 = 6, drawn into 6 and 2 gives 12 (by 6th step) to be set in the 3d col. against 36 in the first : dotting through all the second columns all such multiples, by which the fractions are found. Proceeding thus, the whole number of different fractions are truly determined in a short time.

<center>IV. QUESTION 282, <i>by the Rev.</i> Anth. Baker.</center>

A gentleman would have a silver punch-bowl made in the form of a parabolic conoid, containing exactly two gallons, but being fru-gally inclined, desires first to know what ought to be its inside di-mensions so that, cæteris paribus, it may require the least quantity of silver possible ?

<center>*Answered by Mr. J.* Turner.</center>

When the convex superficies is the least, the conoid must come nearest to the form of a semi-sphere (being under the least surface of all solids of given solidity formed by the rotation of a curved space about its axis) which is when the abscissa and semi-ordinate of the generated parabola are nearly equal : the conoid then being nearly most capacious. Putting $b = 462$ inches in two wine gallons, $p = 3·1416$, $y = $ ordinate, $x = $ abscissa of the parabola. Then the so-lidity of the conoid $= b = \frac{1}{2}pyyx = $ (when $y = x$) $\frac{1}{2}py^3$; where $y = 6·65$ inches. But accurately, $\frac{p}{6a}(aa + 4yy)^{\frac{3}{2}} - \frac{paa}{6}$ is the su-perficial content of the same (see p. 202, of Mr. Emerson's Fluxions). Now if the value of yy be substituted therein, and the fluxion of it made $= 0$, then $a = 6·1$; $y = 6·54$; $x = 6·94$; and the convex sur-face $= 235·5$ square inches.

<center>*The same answered by Mr.* Ash.</center>

The conoid's convex surface is $\frac{p}{3a}(aa + 4yy)^{\frac{3}{2}} - \frac{paa}{3}$ a mini-

mum ($p = 1.5708$); $\therefore$ $\left(\dfrac{aa + 4yy}{a}\right)^{\frac{3}{2}} - aa$ is also a minimum. Put $cc =$ double solidity $= 4$ wine gall. $qq =$ area of a circle whose rad. $= 1$. Then, by the curve, $ax = yy$, and $qqyyx = cc$, whence $4yy = \dfrac{4c\sqrt{a}}{q}$; and substituting for $4yy$ in the minimum, it becomes $\left(\dfrac{qaa + 4c\sqrt{a}}{a}\right)^{\frac{3}{2}} - aa$. Whence by fluxions, $a = \left(\dfrac{15}{24} + \sqrt{\dfrac{11cc}{192qq}}\right)^{\frac{3}{2}}$ $= 6.03$; whence semi-axis $= 6.98$, and semi-ordinate $= 6.49$ maxime.

<div style="text-align:center">V. QUESTION 283, <i>by Mr.</i> N. Farrer.</div>

Quere the axis and parameter of a parabola and semi-ellipsis, when the latter is circumscribed by the former, whose ordinate is equal to the conjugate axis, and abscissa equal to the semi-transverse: both curves having the same focus, the difference of their parameters 2, and the length of the parabolic curve being 28.68?

<div style="text-align:center"><i>Answered by Mr.</i> Ash.</div>

'Tis inconsistent with the nature of the parabola and semi-ellipsis that the latter should be circumscribed by the former, when the ordinate $=$ conjugate, and axis $=$ semi-transverse (which Mr. Turner also observes): $\therefore$ semi-ellipsis must be read for parabola, & contra. Then from the properties of the curve and conditions of the question, the parameter of the parabola is found $= 2$. Suppose AH $= \varsigma = 12.5$; the corresponding curve A$r = 13.90375$ &c. $\therefore$ the remaining part $rm (= z)$ of the curve $= 0.43625$.
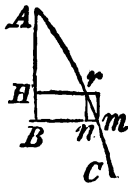
Put $y = nm$, $x = nr$, and we shall have $\dot{y} = \dfrac{a\dot{x}}{\sqrt{(4ac + 4ax)}}$, and $(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} = \dot{z} = \sqrt{(1 + \dfrac{a}{4c + 4x})}\,\dot{x}$; whose fluent is $x + \dfrac{ax}{8c} - \dfrac{aax}{128cc} - \dfrac{axx}{16cc} + \dfrac{aaxx}{128c^3}$ &c. $= z$; or $ax - bx^2 + cx^3$ &c. $= z$ (by substituting a for $1 + \dfrac{a}{8c} - \dfrac{aa}{128cc}$; b for $\dfrac{-a}{16cc} + \dfrac{aa}{128c^3}$; &c.) And by reversion $x = \dfrac{z}{a} + \dfrac{bz}{a^3} + \dfrac{2bb - ac}{a^5}z^3$ &c. $= 0.4289$; which adde to AH makes 12.9289 for the required axis of the parabola.

Additional Solution.

As the method of finding the parameter is not inserted, I shall here supply it, and at the same time give another very easy method of solving the latter part of the question.

Since then $ax = yy$ by the nature of the parabola, putting $y =$ the ordinate, $x =$ the abscissa, and $a =$ the parameter of the parabola; and, by the nature of the ellipse, $x : y :: 2y : 2yy \div x =$ parameter of the ellipse $= a + 2$ by the question; or $ax + 2x = 2yy$; subtract the former equation from this, then $2x = yy = ax$, and therefore $a = 2$ the parameter of the parabola.

The parameter being thus found, by p. 309, of my Mensuration, the length of the double curve will be $y \sqrt{(1 + yy)} + $ hyp. log. of $(y + \sqrt{(1 + yy)}) = 28.68$, by the question, $= c$; call the hyp. log. of $(y + \sqrt{(1 + yy)})$, v; then $y \sqrt{(1 + yy)} = c - v$. Now it is easy to perceive that $y = 5$ nearly; then since a small difference in the value of y will make a still smaller and inconsiderable difference in the value of v, and when y is supposed 5 then $v = 2.312456$, and then the equation $y \sqrt{(1 + yy)} = c - v$ becomes $y \sqrt{(1 + yy)} = 26.367564 = d$, and $y = \sqrt{(\sqrt{(dd + \frac{1}{4})} + \frac{1}{2})} = 5.08$. With this value of y find a new value of $v = 2.3286281$, and thence of $d = 26.3513719$; and this same theorem will give $y = 5.08501$, which is very exact.

Then $x = \dfrac{yy}{a} = \dfrac{yy}{2} = 12.92806$.   H.

VI.   QUESTION 284, *by Mr.* Cuth. Cockson.

Quere with what part of a cylindrical stick should a person strike, to give the greatest blow; the length of the arm being 20 inches, and that of the stick 50?

Answered by Mr. J. Turner.

Put $a = 70$, the length of the arm and stick; $b = 20$ the length of the arm; then the distance of the centre of percussion of the stick from the upper end of the arm will be $= \dfrac{2aa + 2ab + 2bb}{3a + 3b} = $ 49.63 inches, and from the farthest end of the stick $= 20.37$ inches (vide Stone's Flux. p. 177).

The same answered by Mr. J. Ash.

Let $20 = c$ represent the length of the arm, and put $x =$ the distance of the centre of percussion from the hand. Then $(c + xx) \times \dot{x} = $ fluxion of the momentums; and $(c + x)^2 \times \dot{x} = $ that of the forces; the fluent of which divided by that of the momentums is

$\dfrac{6cc + 6cx + 2xx}{6c + 3x} = $ the distance of the centre of percussion of the part xp, from the point of suspension; which (when $x = 50$) will be $= 49\cdot6296$ &c. Consequently the part of the stick required is $20\cdot37037$ &c. inches from the top.

VII. QUESTION 285, by Mr. Landen.

I am about building a house, the breadth whereof I design shall be 24 feet, and the perpendicular height from the ground to the ridge 42 feet. The ends, which are to point directly east and west, will be sheltered by neighbouring tenements; but the sides will be exposed to the fury of the north and south winds. I therefore would be satisfied what the angle of the ridge must be, and how the side walls, that the wind blowing from either of those quarters shall have the least effect on the building?

Answered by Mr. Landen, the Proposer.

Put $a = 42$ the height of the building; $b = 12$, half the breadth; $x = $ perpendicular height of the roof. The force of all the particles of the air impinging against the wall, to blow upon the building, will be as $aa - 2ax + xx$. The effect of any particles striking against the roof (found by mechanics) is to the effect the same would have had, striking directly against an upright plane, as the square of the sine of the angle of incidence to the square of radius; therefore the force of all the particles against the roof, to blow down the building, will be as $\dfrac{2ax^3 - x^4}{bb + xx}$. And by the question $\dfrac{2ax^3 - x^4}{bb + xx} + aa - 2ax + xx$ must be a minimum. Let the fluxion of it be put $= 0$, then $x = \dfrac{b\sqrt{(bb + 4aa)} - bb}{2a} = 10\cdot4076$ feet: Whence the height of the side-walls must be $31\cdot5924$ feet, and the ridge angle $98° 7'$.

Mr. Jepson's Solution is as follows.

Let BR $= 42 = a$, AB $=$ CD $= 12 = b$, CR $= y$, DR $= x$, then AC $=$ BD $= a - x$. By p. 255 of Emerson's Fluxions, the resistance of the plane DR, to the resistance of the plane CR, is as yy to xx (supposing the wind to blow in a direction parallel to the plane of the horizon); but the resistance of the plane DR is $= cx$ (putting $c = $ the length of the building) and (per 47 Euc. I.) $yy = bb + xx$, therefore the resistance of the plane CR is $= \dfrac{cx}{bb + xx}$, and that
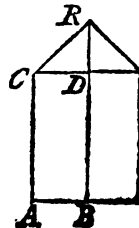
of the side wall $AC = ca - cx$, $\therefore \dfrac{x^3}{bb + xx} + a - x$ is a mini-
mum; which in fluxions, &c. gives $bbxx - b^4 = 0$, and $x = b = 12$. Hence the height of the side walls $= 30$ feet, and ridge angle $90°$, which will always be the same, let the height and length of the building be what they will.

VIII. QUESTION 286, *by* Mr. J. Ash.

If a parabolic conoid, whose altitude is 9, and base 6 inches be cut by a.right line at some distance from, but parallel to its axis, what is the solidity and convex surface of that segment, or part cut off, when the height of the plane of that section is 5 inches?

Answered by the Proposer Mr. Ash.

Let $x = FP$, $n = 2$ $FE = 6$. By the nature of the circle we get $\sqrt{(nx - xx)} = Pm = \sqrt{PA}$ by the nature of the parabola (because 9 the altitude of the whole parabola $= 3^x$ the square of its semi-ordi-nate, and all the perpendicular sections are simi-lar parabolas); and $\frac{4}{3}(nx - xx)^{\frac{3}{4}} =$ area

mAm; consequently $\frac{4}{3}(nx - xx)^{\frac{3}{2}} \dot{x} =$ flux. of
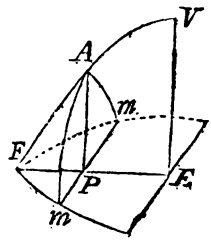
of the solidity; whose fluent is $\dfrac{4\sqrt{nx}}{3} \times :$

$\dfrac{2nx^2}{5} - \dfrac{6x^3}{14} + \dfrac{6x^4}{72n}$ &c. $= 6{\cdot}1122$ the solid content.

Again, since $nx - xx = Pm^2 = AP$, whence the fluxion of the curve is $\dot{x}\sqrt{(nn - 4nx + 4xx + 1)}$, or putting cc for $nn + 1$, $\dot{x}\sqrt{(cc - 4nx + 4xx)}$; and because $2\sqrt{(nx - xx)} =$ chord of the arch mFm, and $\sqrt{(nx)} =$ chord of half that arch; therefore (by Huygen's Theor.) the arch $mFm = \dfrac{8\sqrt{(nx)} - 2\sqrt{(nx - xx)}}{3}$ nearly. Consequently

$\dfrac{8\sqrt{(nx)} - 2\sqrt{(nx - xx)}}{3} \times \dot{x}\sqrt{(cc - 4nx + 4xx)} =$ fluxion of the surface. The fluent of which (when $x = 1$) is (if no mistake be made) $= 16{\cdot}34$ fere.

Remark. It may also be observed that the solidity required in this question, may be found without an infinite series, by the rules at p. 335, &c. of my *Mensuration.* And using the numbers given in this question, the solidity by those rules will become $= 6{\cdot}48510957$ very accurately. H.

IX. QUESTION 287, *by Mr.* Cuth. Cockson.

Given $a^3\ddot{y} + \ddot{y}\dot{y}\dot{x}^2 - a\dot{a}\dot{y}\dot{y} = 0$, the equation of a curve, whose radius of evolution at the vertex is $140 = a$; to find the value of y, the abscissa, when its corresponding semi-ordinate $x = 50$?

Answered by Mr. Emerson.

At p. 39 and 40 of my Fluxions, from whence it was taken; y being there $= \dfrac{x^2}{2a} + \dfrac{x^4}{24a^3} + \dfrac{x^6}{720a^5} - \dfrac{x^8}{4480a^7}$, &c. $= 9{\cdot}0238$ &c.

X. QUESTION 288, *by Mr.* John Hampson.

Required to find three numbers, that when each is severally added to the cube of their sum, their respective sums shall be a cube number?

Answered by Mr. Hampson only.

1. Answer, $a = \frac{23625}{157464}$, $b = \frac{1539}{157464}$, $c = \frac{18577}{157464}$.

2. Answer, $a = \frac{18954}{132651}$, $b = \frac{4184}{132651}$, $c = \frac{271}{132651}$.

Solution from the 5th book of Diophantus, *Quest.* 18.

To find rational values of x, y, z, u, v, w, such that $(x+y+z)^3 + x^3 = u^3$, $(x+y+z)^3 + y^3 = v^3$, $(x+y+z)^3 + z^3 = w^3$, put $x+y+z = n$, $x = 7n^3$, $y = 26n^3$, $z = 63n^3$.

And it is plain that the cube of the sum increased by x, y, or z will make a cube. It remains to make the sum of the three equal to n, or $96n^3 = n$, or $96n^2 = 1$.

Now unity is a square number, and if 96 were also a square the problem would be solved. Now 96 is the sum of three numbers, any one of which increased by unity makes a cube. We are therefore to find three numbers, such that each increased by unity is a cube, and the sum of the three a square.

For the side of the first cube, put $n' + 1$, of the second $2 - n'$, of the third 2. The first cube is $n'^3 + 3n'^2 + 3n' + 1$, the second $-n'^3 + 6n'^2 - 12n' + 8$, the third 8. The numbers therefore are $n'^3 + 3n'^2 + 3n'$, $-n'^3 + 6n'^2 - 12n' + 7$, and 7. And the sum of these $9n'^2 - 9n' + 14$, which therefore must be a square, suppose $(3n' - 4)^2$. Therefore $n' = \frac{2}{15}$. The first sought number is therefore $\frac{1539}{3375}$, second $\frac{18577}{3375}$, third 7. Thus I arrive at what was at first proposed,

and I put the first number $x = \frac{1538}{3375}n^3$, $y = \frac{19577}{3375}n^3$, $z = 7n^3$; and I

again put the sum of all equal to n, therefore $\frac{43740}{3375}n^3 = n$, or $2196n^2$

$= 225$, or $n = \frac{15}{54}$. Ad positiones et constat. L.

Another Additional Solution.

Let the three numbers sought be denoted by $(a^3 - u^6)v^3$, $(b^3 - u^6)v^3$ and $(c^3 - u^6)v^3$, and their sum by u^3v; then if each number be added to the cube of their sum, there results the three cubes a^3v^3, b^3v^3, and c^3v^3; therefore all the conditions of the question will be fulfilled if the sum of the three numbers be $= u^3v$, that is if $(a^3 + b^3 + c^3 - 3u^6)v^3 = u^3v$, or dividing by v, $(a^3+b^3+c^3 - 3u^6)v^2 = u^3$.

Therefore in order that v may be a rational number, it is evident that $a^3 + b^3 + c^3 - 3u^6$ must be a square.

Now it appears by a few trials that this will be the case when $a = 4$, $b = 5$, $c = 7$, and $u = 1$, for then $a^3 + b^3 + c^3 - 3u^6 = 64 + 125 + 343 - 3 = 529 = (23)^2$. Hence $23^2v^2 = u^3 = 1$ and $v = \frac{1}{23}$; therefore the numbers are in this case

$$(a^3 - u^6)v^3 = (64-1) \times \frac{1}{23^3} = \frac{63}{12167}$$

$$(b^3 - u^6)v^3 = (125-1) \times \frac{1}{23^3} = \frac{124}{12167}$$

$$(c^3 - u^6)v^3 = (343-1) \times \frac{1}{23^3} = \frac{342}{12167}.$$

Other numbers that will answer may be found in a similar manner: but we may at once obtain an indefinite number of answers, without trial, by assuming $a = (p + q)v$, $b = (r - p)v$ and $c = sv$, then $a^3 + b^3 + c^3 - 3u^6$ becomes $=$

$$3(q + r)p^2 + 3(q^2 - r^2)p + q^3 + r^3 + s^3 - 3u^6.$$

And to make this a square take $3(q + r) =$ some square number, n^2, then

$n^2p^2 + 3(q^2 - r^2)p + q^3 + r^3 + s^3 - 3u^6 = $ a square $= (m - np)^2$
$= m^2 - 2mnp + n^2p^2$; whence by reduction

$$p = \frac{m^2 + 3u^6 - (q^3 + r^3 + s^3)}{3(q^2 - r^2) + 2mn},$$ where m, q, r, and s may

be taken at pleasure within certain limits easily determinable for any assumed value of u; q and r being so related that three times their sum may be a square number, n^2.

Example. Suppose $u = 1$, $q = 1$, $r = 2$, $s = 2$, and $m = 4$; then $n = 3$ and

$$p = \frac{4^2 + 3 - (1 + 8 - 8)}{3(1 - 4) + 24} = \frac{2}{15}$$

$$v = \frac{u}{m - np} = 1 \div \frac{18}{15} = \frac{5}{18}$$

$a = (p + q)v = \frac{17}{15}v, \; b = (r - p)v = \frac{28}{15}v, \; c = sv = \frac{30}{15}v.$

Hence $(a^3 - u^6)v^3 = \dfrac{17^3 - 15^3}{3^3 \times 18^3} = \dfrac{1538}{157464}$

$(b^3 - u^6)v^3 = \dfrac{28^3 - 15^3}{3^3 \times 18^3} = \dfrac{18577}{157464}$

$(c^3 - u^6)v^3 = \dfrac{30^3 - 15^3}{3^3 \times 18^3} = \dfrac{23625}{157464}.$ L.

XI. QUESTION 289, *by Mr.* Heath.

On what days of the year do our shadows move slowest and fastest in London? And at what times do they move slowest and fastest on any day?

Answered by Mr. Heath.

It is evident, that, when the sun's motion is most vertical, his increase of azimuth is least, and consequently of altitude greatest; and when his motion is most horizontal, his increase of azimuth greatest, and of altitude least.

At 12 each day, the sun's motion is parallel to the horizon, or least vertical, being then perpendicular to the meridian, or vertical circle: therefore the circular motion of shadows, in general, is at noon the fastest.

To find when the increase of azimuth is least on any day, and in any place.

Put $p = $ s. ⊙p, the sun's distance from the pole; $d = $ its cos. or sine of declination; $c = $ s. zp; $b = $ its cos or s. of latitude; $y = $ s. azim. ⊙zp. Now if r⊙om be the semi-diurnal arc of a lesser circle, it is evident that the sun's motion is most vertical when angle o⊙z is least, or its cos. z⊙p greatest (the lesser circle coinciding, at the point ⊙, with a greater, at right angles with ⊙p); say, $p : y :: c :$ $cy \div p = $ a maximum, which is evidently when $y = 1$, the sine of 90°. Whence, azimuth increases slowest and altitude fastest when the sun is due east, on all days, and in all places whatsoever; and therefore shadows in general move slowest when the sun is due east; or at sun-rise (the most easterly) in the southern declinations.
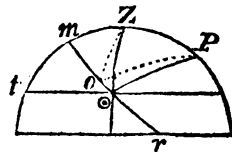
In the maximum $cy \div p$, when p is least (the sine of 66° 30' or 113° 30') the azimuth on those days increases the slowest of all other days, and altitude fastest: Whence, our shadows move slowest of all on the longest and shortest days of the year: the sun due east, or at rising. But when $p = 1$, the increase of azimuth, on that day, is least slow (at sun due E.) and the increase of altitude the least fast; and as the azimuth increases its swiftness till noon, therefore

our shadows move fastest of all at noon, on the day of the year when the sun is in the equinoctial.

By erecting a wire perpendicular to an horizontal plane, and marking the circular increase of the shadow at equal distances of time, the truth of the foregoing will appear by inspection. And by this circular motion, and the shortening one of the shadow, or by the increase of azimuth and altitude together, the summit of it is made to describe a curve, whose nature is determinable according to the latitude of the place and sun's declination.

N. B. The increase of azimuth being always as the angle $\vee\odot z$, and the increase of altitude as its comp. the $\angle o \odot t$, the sum of both increases (viz. of azim. and alt.) will be at all times and all places alike; *i. e.* the sum of the circular and shortening motions of shadows on a horizontal plane will be every where equal.

The increase of an azimuth is but the fluxion of it, and the slowest increase of an azimuth is but the least fluxion of that fluxion of azimuth; therefore if the angle of time be substituted for (which flows equally) and an expression be raised of the azimuth angle (by trigon. and series) the fluxion of that fluxion made $= 0$ will exhibit an equation, shewing the azimuth properties as aforesaid.

Otherwise by the Rev. Mr. Cha. Wildbore, from Dr. Hutton's Edition of the Diaries.

'Tis manifest from my solution to the 654th diary question, that at all places within the polar circles, the velocity of the shadow of the summit of an object, increases from passing due north (or south in the Antartic) till it attains the maximum there determined, after which the velocity continually decreases till noon, when it is the least possible. And with regard to the different greatest velocity on different days, 'tis evident that the expression for the maximum $PE^2 \times PE \times rs + Pb - rP \times PE \times rs - Pb + rP$ will continually increase, by increasing rs or diminishing the declination; consequently the greatest velocity on the longest day will be less than the maximum on any other day : the said maximum increasing till its time and that of rising coincide, when it will be infinite

There is likewise a time in problems of this kind, when the velocity of the shadow, after being the greatest, decreases the fastest which may be found by taking the second fluxion of the above maximum $= 0$. But in our latitude both these times coincide at sunrising, when the velocity being always infinite, decreases, and that more and more slowly till noon, when it is always the least possible. And the same may much more be said of the velocity with which the altitude alone increases. As to the velocity with which the azimuth alone increases, what Mr. Simpson has done upon the subject is right; for it must increase the slowest when the fluxion of the hour angle P bears the greatest ratio possible to that of the azimuth z, the reason for which is no more, than if any small distance $\dot{x}$ be divided

by the time $\dot{x} \div v$ in which it is gone over, it will give v the ve-
locity of uniform motion at that time. Therefore the fluxion of the
azimuth, divided by that of the time, must give the velocity with
which the azimuth increases at that time, or $\dfrac{\cos \cdot \odot}{s \cdot z\odot} =$ (vide my

solution above referred to) $\dfrac{\text{P}b \cdot \text{PE} - r\text{P}}{r\text{s} \cdot \text{EO} \cdot ao}$ or $\left(\text{because } ao = \dfrac{\text{EO}}{\text{EP}}\right)$

$\dfrac{\text{P}b \cdot \text{PE}^2 - r\text{P} \cdot \text{PE}}{\text{PE}^2 - 1}$ or $\dfrac{\text{P}b - r\text{P} \cdot \text{PE}}{\text{PE}^2 - 1}$ is a minimum. Which will evi-
dently be less, the greater $r\text{P}$ the sine of declination is taken; and
therefore the longer the days are, the less will the slowest increase
of azimuth be, and consequently it will be less on the longest day
than on any other day. Moreover, when the sun is in the equinoc-
tial, the minimum becoming $\dfrac{\text{P}b}{\text{PE}^2 - 1}$ will be less as PE is greater,

and therefore will be the least possible when PE is greatest, and the
altitude $= 0$, or at sun-rising; therefore on the equinoctial days the
azimuth will increase the slowest at rising due east. Consequently
when the declination is south, the minimum will be before sun-rising,
and therefore in that case the azimuth will increase faster and faster
from sun-rising till noon, when it will in all cases increase the fastest,
but on the shortest day slower than on any other day, and on the
longest faster.

<center>XII.   QUESTION 290, <i>by Mr.</i> Bulman.</center>

Being at a town in Kent, I observed three objects on the other
side of the river Medway, (a castle, wind-mill, and spire) whose
distances from one another are known : From the castle (the nearest
object seen) to the spire, is 10 furlongs ; from the castle to the wind-
mill 23 furlongs ; and from the wind-mill to the spire is 25 furlongs.
I also observed the town angle between the castle and spire $= 28°$
$34'$, and the town angle between the castle and wind-mill $= 57° 45'$.
What distance did I stand from each of those objects ? And give a
geometrical construction of the same.

<center><i>Answered by Mr. J. Ash.</i></center>

The $\triangle$ csw being formed, make $\angle$ wsd $=$
$57° 45'$, and $\angle$ dws $= 28° 34'$; then circum-
scribe $\triangle$ swd with a circle : draw dt through
c, and the point t is the town's situation.
From whence by trigonometry, tc $= 0.65707$,
tw $= 23.3439$, and ts $= 10.5721$ fere.
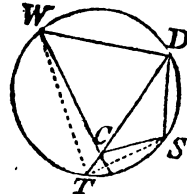
See also question 100.

XIII. QUESTION 291, *by Mr. J.* Ash.

A lady of important speculation,
Would gladly know her age from this equation.

$$1 \cdot 05^{A} = \frac{\sqrt{A}}{1 - \cdot 05\sqrt{A}}.$$

Answered by Mr. Ash.

By trials A is found between 48 and 49; put $(7 - x)^2 = A$, $1 \cdot 05^{(7-x)^2} = \dfrac{7 - x}{0 \cdot 65 + 0 \cdot 5x}$ per quest. (which substitution is neces-sary to make the following series converge). Let $c = 0 \cdot 5$, $n = 7$, $q = $ hyp. log. 7, $m = \cdot 65$, and $p = $ its log. also $a = $ log. $1 \cdot 05$; then (by log. series) $(n - x)^2 a = q - \dfrac{x}{n} - \dfrac{x^2}{2n^2} - \dfrac{x^3}{3n^3}$ &c. $+ p - \dfrac{cx}{m}$ $- \dfrac{ccxx}{2mm} - \dfrac{c^3 x^3}{3m^3}$ &c. And by reversion, $x = 0 \cdot 030409$; $\therefore (n - x)^2$ $= A = 48 \cdot 5752$ &c. the lady's age.

XIV. QUESTION 292, *by Mr.* Heath.

Suppose a person has a legacy left him, such, that, the pounds power of Napier's logarithm thereof, is equal to the shillings in the legacy raised to the power denoted by Napier's logarithm of the pounds: what is the amount of the legacy?

Answered by Mr. J. Turner.

Put $x = $ the pounds of the legacy; then $(1. x)^x = (20x)^{1. x}$ per question; or putting $b = 2 \cdot 30258$ &c. $(b\, 1.\, x)^x = (20x)^{b\, 1.\, x}$ by common logarithms: And by a few trials $x = 15 \cdot 615 = 15l.\ 12s.\ 3d.$ required.

XV. QUESTION 293, *by Mr.* Bulman.

A spheroidal ullage lies upon the ground with the bung uppermost, from whence to the surface of the liquor (which is exactly the height of the upper ends of the cask) is 9 inches, and its diagonal either way from the bung to the lower ends $= 55$ inches, its ullage is a maximum: Quere the content of the cask and ullage, brother gaugers?

Answered by Mr. Heath

As the contained liquor always bears a proportion considerably greater than the vacuity, in the several variations of the form of the cask (which supposing but 2 to 1) the greater the vacuity, the greater will be the quantity of liquor, and consequently the sum of both; therefore when the ullage is a maximum, the whole cask will be also a maximum.
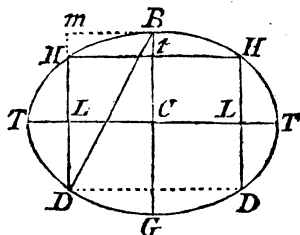
First, *to find the content of spheroidal frustums.* Let $t = $ TT, the transverse; $b = $ BG, the conjugate of the generating ellipsis; and $x = $ CL. By the property of the curve, $tt : bb :: \dfrac{tt}{4} - xx : \dfrac{bb}{4} - \dfrac{bbxx}{tt} = $ HL^2; consequently $bb - \dfrac{4bbxx}{tt} = $ HD$^2 = hh$, (whence $tt = \dfrac{4bbxx}{bb - hh}$), and $n\dot{x} \times \dfrac{bbtt - 4bbxx}{tt} = $ flux. frust. BGDHB (n being ·7854) whose fluent is $*nx \times \dfrac{3bbtt - 4bbxx}{3tt}$; in which substituting for tt, we have $nx \times \dfrac{2bb + hh}{3} = $ the content BGDHB, which holds true when it is the segment or frustum of a sphere.

Theorem. Multiply the length of any spheroidal cask into the sum of twice the square of the bung diameter added to the square of the head diameter, and that product multiplied by ·2618 will give the content in inches; or divided instead by 1077·158, or 882·3529 will give the content in ale or wine gallons: or multiply instead by ·00092837, or ·0011333, &c.

N. B. If x be put for TL; then the flux. segment DTH $= n\dot{x} \times \dfrac{bbtx - bbxx}{tt}$, whose fluent gives the same theorem as above for the content.

Now if $c = $ Bt, $d = $ BD, and x be put $= $ Dm; then $x - c = $ head diam. $x + c = $ bung diam. $2\sqrt{(dd - xx)} = $ length of the cask and $\dfrac{3cc + 2cx + 3xx}{\text{Divisor}} \sqrt{(dd - xx)} = $ the content, a maximum. In fluxions and reduced, $9x^3 + 4cxx + 3ccx = 6ddx + 2cdd$, where $x = 44·1615$. Whence head diam. $= 35·1615$, bung diam. $= 53·1615$, length $= 65·0567$ inches, and content $= 419·313$ ale gallons. Again, put $x = $ Bt; the section or surface of ullage, HtH, is an ellipsis, whose conjugate

$= 2\sqrt{(bx - xx)}$; and tc being $\dfrac{b}{2} - x = $ HL, if $y = $ LC, then tt :

$bb :: \dfrac{tt}{4} - yy : \dfrac{bb}{4} - bx + xx$, and $2y = \dfrac{2t}{b}\sqrt{(bx - xx)} =$

transverse axis. The area section is $4nt \times \dfrac{bx - xx}{b}$, and flux.

segment HBH, $4nt\dot{x} \times \dfrac{bx - xx}{b}$, whose fluent $4ntx \times \dfrac{3b - 2x}{6b}$ is its

content (which when $b = t$ becomes $4nxx \times \dfrac{3t - 2x}{6}$ for the sphe-

rical segment, as the other segment * becomes $nx \times \dfrac{3tt - 4xx}{3}$,

proved also in Diary, p. 45, 1747). But, by above $t = \dfrac{2bx}{\sqrt{(bb - hh)}}$

$= 87\cdot4207$; whence the content of the frustum or segment HBH $=$
$34\cdot99$ gallons; consequently $384\cdot322$ ale gallons of liquor remain in
the cask.

The principle of the above solution is not accurately true, but only
an approximation, as will appear from the following

Solution by the Rev. Mr. Wildbore, taken from the Appendix to Dr. Hutton's Edition of the Diaries.

In the original solution to this question, it is said that the greater
the vacuity, the greater will be the quantity of liquor, and conse-
quently the sum of both; therefore when the ullage is a max. the
whole cask will be also a max. But this method of reasoning is not
altogether to be depended on : for retaining the notation there given,
viz. $x = $ DM $= $ GT, $d = $ BD, and $c = $ BT; then the conjugate semi-
axis of the surface of the liquor, being the semi-ordinate passing
through t of the circle whose diam. is BG, must be $= \sqrt{cx}$; whence,
$(x - c$ being the head, $x + c$ the bung diam. $2\sqrt{(dd - xx)} = $ the
length of the cask, and $p = 3\cdot14159$) by Hutton's Mensuration, pa.
278 and 287, we have $\frac{1}{6}p \times (3xx + 2cx + 3aa)\sqrt{(dd - xx)} = $ the

content of the whole cask, and $\frac{1}{6}p \times \dfrac{3ccx + c^3}{\sqrt{cx}}\sqrt{(dd - xx)} = $ that

of the empty part; consequently the diff. of these two must be the
ullage.

But these expressions are so very different, that it is evident at
first sight that the one will not be a max. when the other is so; and
consequently their diff. cannot be a max. when either of them is so.
But the question expressly requires the ullage to be a max. and
though this, by reason of the smallness of the vacuity in this par-
ticular case, be nearly a max. when the cask is a max. it will
not be exactly so; and cases may be put in which they will be very

different: and in the present, though there be but little difference in the content, the form of the cask will come out considerably different, when the ullage is a max. from what it does when the cask is so. Thus, putting $cx = vv$, when the ullage is a max. then $\left(\dfrac{3v^4}{cc}\right.$

$+ 2vv + 3co - 3cv - \dfrac{c^3}{v}\right) \sqrt{\left(dd - \dfrac{v^3}{cc}\right)}$ is so; the fluxion, reduced gives $v^9 + \dfrac{4c^2v^7}{9} - \dfrac{c^3v^6}{2} - \dfrac{2d^2c^2 - c^4}{3}v^5 - \dfrac{c^5v^4}{18} - \dfrac{2d^2c^4v^3}{9}$

$+ \dfrac{d^2c^5v^2}{6} - \dfrac{c^7d^2}{18} = 0$; in numbers, $v^9 + 36v^7 - 364\frac{1}{2}v^6 - 161163v^5$

$- 3280\frac{1}{2}v^4 - 4410450v^3 + 3307837\frac{1}{2}v^2 - 89356612\frac{1}{2} = 0$. Whence $v = 20\cdot1424$; $x = 45\cdot07958$, head diam. $= 36\cdot07958$, bung. diam. $54\cdot07958$, length $= 63\cdot01844$, and ullage $= 384\cdot4266$ ale gallons.

But if the vacuity be a max. the form of the cask, will be still more different, and the content still less. For $(3x + c) \sqrt{\left(\dfrac{dd}{x} - x\right)}$

being a max. its fluxion, reduced, is $3ddx - 9x^3 - cd^2 - cx^2 = 0$; in which case therefore x is only $= 29\cdot625$.

XVI. QUESTION 294, *by Mr.* Chr. Mason.

In an evening lately, hearing the noise of guns at a small distance off at sea, I straight repaired to the strand, where I loomed a man of war's tender giving chace to a French privateer; I perceived a flash of a gun from the tender S.W. by S. at 33 half seconds of time heard the report: Eight minutes after I observed another flash at S.S.W. from the same, and at 25 half seconds heard the report. The chase kept her course, making equal way; in 35 minutes more was drove on shore, being but two minutes of time a-head of the tender. What was the bearing and distance between my station, the tender, and the pilfering poltroon when stranded?

Answered by Mr. T. Cowper.

Let D and T be the places of the tender at the first and second observations; t and P the places of the tender and privateer when the latter is stranded; and s the station of the observer. By the quest. $\angle TSD = 11° 15'$, and sound moving 1142 feet per second, SD $= 18843$ feet, and ST $= 14275$; whence by trigon. TD $= 5586$, and $\angle$STO $= 41° 9\frac{1}{4}'$. Now the interval between the first and second flashes at
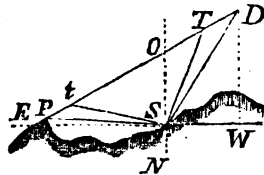
D, and T, being 8' 16½'', and from the flash at T, till the privateer was drove ashore at P, by the tender at t, 35' 12½''; by proportion of motion, Tt will be found 23767·22, and tP = 1350·09 feet. Whence $\angle ost = 80° 31'$, the bearing of the tender about E. by S. dist. st = 3·04061 miles; also the $\angle osP = 84° 10'$, the privateer's bearing about E. ½ S. dist. sP = 3·2514 miles required.

THE PRIZE QUESTION, *by Mr.* Bamfield.

If the diameter of Sysiphus's cylindrical stone be two feet, which he continually rolls upon the surface of a semi-globular mountain, half a mile high : Quere what space will a spot on the convex surface of that stone travel through in rolling directly up and down the said mountain? And what will be the time of its descent from the top by the force of gravity?

Answered by Mr. J. Landen.

Let Md be perpendicular to the epicycloid AMm, and *md* another perpendicular supposed infinitely near the former; then the concentric arches M*b*, *m*c, being described from the centre D of the immovable circle; the chords c*b*, cc of the moveable one will be respectively = Mo, mn.
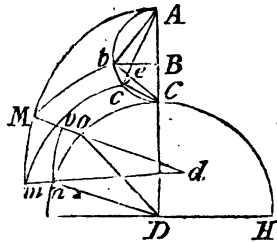
From the centre d (the point where the perpendiculars Md, md touch the evolute) describe the small arch nv; and the small right lines be, ce, being considered as small arches, described from their respective centres A and c; the small right-angled triangles bce, vno, will be similar and equal; bc being = no, and ce = nv; ∴ cc : dn : : $\angle ndv$: cce. But ndv = nDo+cce. And calling AC, $a = 2$; CD, $b = 2640$ feet; AB, x; we get $nd = \frac{b\sqrt{(aa-ax)}}{a+b}$; now, as $\frac{b\sqrt{(aa-ax)}}{a+b}$ (nd) : $\frac{ax}{2\sqrt{ax}}$ (vn = ce) :: $\frac{a+2b}{a+b} \sqrt{(aa-ax)}$ (mn + nd) : $\frac{a+2b}{2b\sqrt{ax}} \times ax$ (Mm) the fluxion

of the epicycloid; the fluent whereof is $\frac{a+2b}{b}\sqrt{ax}$ = half the space

a spot on the convex surface of the stone will go through in one revolution (when $x = a = 2$) = 4·001515, &c. Consequently 8·00303, &c. feet is the whole space described at each revolution; and the stone revolving exactly 1320 times, going up and down the mountain, the whole space gone through by the spot will be 10564 feet, or 2 00075 &c. miles.

Removing the stone to the next point from the vertex, the gravity will make it descend along the side of the mountain, till its velocity, in an horizontal direction, is the greatest it can acquire; when it will fly off, and descend to the bottom, in the curve of a parabola. Putting $a = 2640$ feet, the mountain's height, $x =$ dist. perp. descended, $s = 16\frac{1}{12}$ feet, a space perp. descended in the 1st second of time, then $\sqrt{s} : 2s :: \sqrt{x} : 2\sqrt{sx}$ the velocity in the curve, which is to the velocity in an horizontal direction, as the tangent is to the ordinate, i. e. $\dfrac{a}{a-x} : 1 :: 2\sqrt{sx} : \dfrac{2\sqrt{s}}{a} \sqrt{(aax -} 2axx + x^3)$ the velocity in a direction parallel to the horizon, whose fluxion reduced is $xx - \dfrac{4ax}{3} + \dfrac{aa}{3} = 0$, where $x = \frac{1}{3}a$ the distance descended when the stone ceases to touch the mountain.

The velocity $2\sqrt{sx}$ applied to $\dfrac{a\dot{x}}{\sqrt{(2ax - xx)}}$, the fluxion of the arc, gives $\dfrac{a}{2\sqrt{s}} \times \dfrac{\dot{x}}{2\sqrt{(2a-x)}}$, the fluxion of time, whose fluent is $\dfrac{\sqrt{a}}{2\sqrt{2s}} \times$ hyp. log. $x + \dfrac{\sqrt{a}}{2\sqrt{2s}} \times : \dfrac{x}{2.2.a} + \dfrac{3x^2}{2.2.4.2^2a^2} + \dfrac{3.5x^3}{3.2.4.6.2^3a^3}$ &c. $= 31''$ (when $x = \frac{1}{3}a$). If to which $6'' 18'''$, the time of descending through the parabolical arch, be added, the sum is $37'' 18'''$, the whole time of descent required; and the stone will fall 326 feet from the foot of the mountain.

Mr. Stone observes in his Mathematical Dictionary, that " as semi-diam. resting circle : sum diameters of both circles :: double sine of half the arc, touching the circle at rest : length of the part of the epicycloid described by a point in the revolving circle ; if upon the convexity of the resting circle. But if upon the concavity ; as semi-diam. resting circle, to the difference of diameters of the touching circle."

In a semi-revolution, 180° of the moving circle touched the circle at rest; the double sine of its half arc $= 2$ (its rad. being unity); $\therefore$ 2640 : 5282 :: 2 : 4·001515, &c. $=$ half the epicycloidal curve, as before.

THE END OF THE FIRST VOLUME.

OCT 20 1921